

Upper Semicontinuity of Attractors for Approximations of Semigroups and Partial Differential Equations

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Abstract. Suppose a given evolutionary equation has a compact attractor and the evolutionary equation is approximated by a finite-dimensional system. Conditions are given to ensure the approximate system has a compact attractor which converges to the original one as the approximation is refined. Applications are given to parabolic and hyperbolic partial differential equations.

1. Introduction. Suppose X is a Banach space and $T(t)$, $t \geq 0$, is a C^r -semigroup on X with $r \geq 0$; that is, $T(t)$, $t \geq 0$, is a semigroup with $T(t)$ continuous in t, x together with the derivatives in x up through the order r .

Following standard terminology (see, for instance, Hale [12]), a set $B \subset X$ is said to *attract* a set $C \subset X$ under the semigroup $T(t)$ if, for any $\varepsilon > 0$, there is a $t_0 = t_0(B, C, \varepsilon)$ such that $T(t)C \subset \mathcal{N}(B, \varepsilon)$ for $t \geq t_0$, where $\mathcal{N}(B, \varepsilon)$ denotes the ε -neighborhood of B . A compact invariant set \mathcal{A} is said to be a *local attractor* if there exists an open neighborhood U of \mathcal{A} such that \mathcal{A} attracts U . The set \mathcal{A} is an *attractor* if, for any bounded set B in X , \mathcal{A} attracts B . Conditions for the existence of an attractor may be found in Hale [12].

Now suppose the semigroup depends on a parameter λ belonging to an open subset of a Banach space, say $T(t) = T_\lambda(t)$, where $T_\lambda(t)x$ is continuous in (t, x, λ) , the continuity in λ being uniform on bounded sets. If \mathcal{A}_{λ_0} is a local attractor for $T_{\lambda_0}(t)$, then additional smoothing properties of $T_\lambda(t)$ will imply there is a neighborhood V of λ_0 such that $T_\lambda(t)$, $\lambda \in V$, has a local attractor \mathcal{A}_λ and \mathcal{A}_λ is *upper semicontinuous* at λ_0 , that is, $\delta_X(\mathcal{A}_\lambda, \mathcal{A}_{\lambda_0}) \rightarrow 0$ as $\lambda \rightarrow \lambda_0$ where, for any two subsets A, B of X ,

$$\delta_X(A, B) = \sup_{x \in A} \text{dist}_X(x, B) \quad \text{and} \quad \text{dist}_X(x, B) = \inf_{x \in B} \|x - y\|_X.$$

The most general result of this type is due to Cooperman [7] and may be found also in Hale [11]. The result for gradient systems is in Hale [12].

The spirit of this paper relates to the above property of upper semicontinuity of a local attractor. Here we consider semigroups $T_h(t)$ depending on a parameter $h > 0$ which "approximate" the semigroup $T(t)$ and give conditions under which there exists a local attractor \mathcal{A}_h for $T_h(t)$ with the property that $\delta_X(\mathcal{A}_h, \mathcal{A}) \rightarrow 0$ as $h \rightarrow 0$. The essential difference between the results here and the ones mentioned before

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is that the approximate semigroups can correspond to Galerkin approximations, splines or discretizations in time of evolutionary equations. These approximations have no uniform continuity property with respect to h .

The outline of the paper is as follows. In Section 2 we give a general approximation result which attempts to bring out the essential elements of the approximate and exact semigroups to ensure that there is a local, compact attractor which is upper semicontinuous. We also give one result in which we assume the approximate semigroups have a local compact attractor and then infer that the exact semigroup has a compact attractor. For the Navier-Stokes equation and the case in which the local attractor for each approximation is a point, Constantin, Foias and Temam [6] have given conditions which ensure that the original equations have an equilibrium. Schmitt, Thompson and Walter [31] discuss the solution of an elliptic boundary value problem in an infinite strip by analyzing solutions of approximate differential equations. This aspect of the problem is important but much more difficult and will be developed further in subsequent publications. The remainder of the paper is devoted to giving specific approximation schemes for particular evolutionary systems for which the hypotheses of Section 2 are satisfied. These applications include spectral projection methods for sectorial evolutionary equations and Galerkin approximations for parabolic equations as well as discretizations in time. Some results about the approximation of the Navier-Stokes equations and of a damped hyperbolic wave equation are also given.

In this paper the convergence of the attractor \mathcal{A}_h to \mathcal{A} as $h \rightarrow 0$ is considered only in the sense of sets. The relationship between the dynamics on the attractors also must be discussed. This problem is much more difficult and requires some knowledge of the flow on \mathcal{A} . Some results on the case in which the flow on \mathcal{A} is Morse-Smale have already been obtained and will appear in Lin and Raugel [25]. For the case of a scalar parabolic equation in one space dimension with a cubic nonlinearity, this latter property has been discussed for space approximation using the Conley index (Khalsa [22]). Numerical computations using Galerkin approximations have been done for a similar example (Mora [28], Rutkowski [30]).

2. A General Approximation Result. In this section we give a general result on the approximation of a local attractor by “approximate” semigroups. These results are very similar to local versions of the ones of Cooperman [7] or Hale [11]. More precisely, let $h > 0$ be a parameter which will tend to 0 and let $(X_h)_h$ be a family of subspaces of X such that

$$(2.1) \quad \lim_{h \rightarrow 0} \text{dist}_X(x, X_h) = 0 \quad \text{for any } x \text{ in } X.$$

Let $T_h(t), t \geq 0$, be a C^s -semigroup on X_h with $s \geq 0$. Actually, $T_h(t)x_h$ need not be a priori defined for all $t > 0$. More precisely, we shall only assume that $T_h(0) = \text{Id}_{X_h}$, $T_h(t+s)x_h = T_h(s)T_h(t)x_h$ for $s \geq 0, t \geq 0$ (as soon as $T_h(t+s)x_h$ and $T_h(s)T_h(t)x_h$ are well defined), that $T_h(t)x_h$ is continuous in t and x_h when it is defined and finally, that $T_h(t)x_h$ is left-continuous at t_1 if $T_h(t)x_h$ exists on $[t_0, t_1]$. The semigroups $T_h(t)$ are said to *conditionally approximate* $T(t)$ on a set $U \subset X$ uniformly on an interval $I \equiv [t_0, t_1] \subset \mathbf{R}^+$ if there are a constant $h(I, U) > 0$ and

a function $\eta(h, I, U)$ defined for $0 < h \leq h(I, U)$ such that

$$(2.2) \quad \lim_{h \rightarrow 0} \eta(h, I, U) = 0$$

and, for any $0 < h \leq h(I, U)$, if $u \in U \cap X_h$ has the property that $T(t)u, T_h(t)u$ are defined and belong to U for $t \in [0, t_2]$ where $t_0 < t_2 \leq t_1$, then

$$(2.3) \quad \|T(t)u - T_h(t)u\|_X \leq \eta(h, I, U) \quad \text{for } t_0 \leq t \leq t_2.$$

The semigroups $T_h(t)$ are said to *approximate* $T(t)$ on a set $U \subset X$ uniformly on an interval $I \subset \mathbf{R}^+$ if $T_h(t)$ conditionally approximates $T(t)$ on U uniformly on I and if, moreover, for $0 < h \leq h(I, U)$ and any $u \in U \cap X_h$, the functions $T(t)u, T_h(t)u$ are defined and satisfy the inequality (2.3) for all $t \in I$.

The semigroups $T_h(t)$ are said to (*conditionally*) *approximate* $T(t)$ on $U \subset X$ uniformly on compact sets of \mathbf{R}^+ if $T_h(t)$ (*conditionally*) approximates $T(t)$ on U uniformly on any compact interval $I \subset \mathbf{R}^+$. We recall that, in the following, $\mathcal{N}(B, \varepsilon)$ denotes the ε -neighborhood of a set B in the Banach space X .

LEMMA 2.1. *Assume that there exist a bounded set $B_0 \subset X$ and an open set $U_0 \supset \mathcal{N}(B_0, d_0)$ for some $d_0 > 0$ such that B_0 attracts U_0 under $T(t)$. Moreover, assume that there exist an open set $U_1 \supset \mathcal{N}(B_0, d_1)$ for some $d_1 > 0$ and a constant $t_0 \geq 0$ such that $T_h(t)$ approximates $T(t)$ on U_1 uniformly on compact sets of $[t_0, \infty)$. Then, for any $\varepsilon_0 > 0$, there are $h_0 > 0$ and $\tau_0 > t_0$ such that, for $0 < h \leq h_0$, for $t \geq \tau_0$,*

$$T_h(t)(U_0 \cap U_1 \cap X_h) \subset \mathcal{N}(B_0, \varepsilon_0).$$

Proof. Without any restriction, we can assume that $\varepsilon_0 \leq \inf(d_0, d_1)$. As B_0 attracts U_0 , there exists $\tau_0 > t_0$ such that, for $t \geq \tau_0$, $T(t)U_0 \subset \mathcal{N}(B_0, \varepsilon_0/2)$. Thanks to the hypothesis (2.2), there exists $h_0 > 0$ such that, for $h \leq h_0$, $\eta(h, [t_0, 2\tau_0], U_1) \leq \varepsilon_0/2$. Therefore, for $h \leq h_0$, for $\tau_0 \leq t \leq 2\tau_0$, $T_h(t)(U_0 \cap U_1 \cap X_h) \subset \mathcal{N}(B_0, \varepsilon_0)$. Let us remark that $U_0 \cap U_1 \cap X_h \neq \emptyset$, because $U_0 \cap U_1 \supset \mathcal{N}(B_0, \inf(d_0, d_1))$.

Now, let us prove by induction that, for $t \geq \tau_0$, $T_h(t)(U_0 \cap U_1 \cap X_h) \subset \mathcal{N}(B_0, \varepsilon_0)$. Assume that, for $\tau_0 \leq t \leq n\tau_0$, $T_h(t)(U_0 \cap U_1 \cap X_h) \subset \mathcal{N}(B_0, \varepsilon_0)$ and let us prove this property for $\tau_0 \leq t \leq (n+1)\tau_0$. If $n\tau_0 \leq t \leq (n+1)\tau_0$, then $t = (n-1)\tau_0 + \tau$ with $\tau_0 \leq \tau \leq 2\tau_0$. Let $u_{0h} \in U_0 \cap U_1 \cap X_h$; we have

$$T_h(t)u_{0h} = T_h(\tau)T_h((n-1)\tau_0)u_{0h}.$$

By the induction hypothesis, $T_h((n-1)\tau_0)u_{0h} \in \mathcal{N}(B_0, \varepsilon_0) \cap X_h$, and hence, $T_h((n-1)\tau_0)u_{0h} \in U_0 \cap U_1 \cap X_h$. Therefore, on the one hand,

$$T(\tau)T_h((n-1)\tau_0)u_{0h} \in \mathcal{N}(B_0, \varepsilon_0/2),$$

and, on the other hand,

$$\|T(\tau)T_h((n-1)\tau_0)u_{0h} - T_h(\tau)T_h((n-1)\tau_0)u_{0h}\|_X \leq \varepsilon_0/2.$$

Finally, $T_h(\tau)T_h((n-1)\tau_0)u_{0h} \in \mathcal{N}(B_0, \varepsilon_0)$, for $\tau_0 \leq \tau \leq 2\tau_0$, i.e., $T_h(t)u_{0h} \in \mathcal{N}(B_0, \varepsilon_0)$ for $\tau_0 \leq t \leq (n+1)\tau_0$. \square

If the dynamical system $T(t)$ has a local compact attractor \mathcal{A} , the hypotheses of Lemma 2.1 can be weakened, as we shall see below.

PROPOSITION 2.2. *Assume that there exist a compact set $\mathcal{A} \subset X$ and an open neighborhood N_1 of \mathcal{A} such that \mathcal{A} attracts N_1 . Suppose that there are constants $h_0 > 0$, $\delta_0 > 0$, $t_0 \geq 0$ and two open neighborhoods N_2, N_3 of \mathcal{A} , with $N_1 \subset N_2 \subset \mathcal{N}(N_2, \delta_0) \subset N_3$, such that, for $0 < h \leq h_0$,*

- (i) $T(t)N_1 \subset N_2$ for $t \geq 0$,
- (ii) $T_h(t)(N_1 \cap X_h) \subset N_2$ for $0 \leq t \leq t_0$,
- (iii) for any $x_h \in \mathcal{N}(N_2, \delta_0) \cap X_h$, there exists $t(x_h) > 0$ such that $T_h(t)x_h \in N_3$ for $0 \leq t \leq t(x_h)$.

Also assume that $T_h(t)$ conditionally approximates $T(t)$ on N_3 uniformly on compact sets of $[t_0, \infty)$. Then, for any $\varepsilon_0 > 0$, there are $\bar{h} > 0$ and $\tau_0 > t_0$ such that, for $0 < h \leq \bar{h}$ and $t \geq \tau_0$,

$$(2.4) \quad T_h(t)(N_1 \cap X_h) \subset \mathcal{N}(A, \varepsilon_0).$$

Proof. As $T_h(t)$ conditionally approximates $T(t)$ on N_3 uniformly on compact sets of $[t_0, +\infty)$, for any $t_1 > t_0$, there is a positive number $\bar{h}(t_1)$ so that $\eta(h, [t_0, t_1], N_3) < \delta_0/4$ for $h \leq \bar{h}(t_1)$. For any $x_h \in N_1 \cap X_h$ and any t , $t_0 \leq t \leq t_1$, we want to prove that $T_h(t)x_h \in N_3$, because this will show that

$$\|T(t)x_h - T_h(t)x_h\|_X \leq \eta(h, [t_0, t_1], N_3)$$

for $t_0 \leq t \leq t_1$ and we may apply Lemma 2.1. Assume this is not the case. Then, by (ii) and (iii), there exists t_2 , $t_0 < t_2 \leq t_1$, such that $T_h(t)x_h \in N_3$ for $0 \leq t < t_2$ and $T_h(t_2)x_h \notin N_3$. But then $T_h(t)x_h \in \mathcal{N}(N_2, \delta_0/4)$ for $0 \leq t < t_2$ and hence $T_h(t_2)x_h \in \mathcal{N}(N_2, \delta_0/2)$, which is a contradiction. This proves the proposition. \square

Remark 2.3. If \mathcal{A} is a local, compact attractor under the semigroup $T(t)$, then \mathcal{A} is stable, and there always exist neighborhoods N_1, N_2 satisfying (i) in Proposition 2.2.

To state the next result, we need some additional terminology. Following Hale, LaSalle and Slemrod [13] or Hale and Lopes [14], a semigroup $T(t)$, $t \geq 0$, on a Banach space X is said to be *asymptotically smooth* if, for any bounded set $B \subset X$, there is a compact set $J = J(B) \subset X$ such that J attracts the set $\{x \in B: T(t)x \in B \text{ for } t \geq 0\}$. A special case of asymptotically smooth semigroups are α -contracting semigroups (see Hale and Lopes [14]); $T(t)$ is an α -contracting semigroup if $T(t) = S(t) + U(t)$, where $U(t)$, $t > 0$, is completely continuous and $S(t)$, $t \geq 0$, is a bounded linear operator for which there is a $\beta > 0$ such that $\|S(t)\|_{\mathcal{L}(X; X)} \leq \exp(-\beta t)$, $t \geq 0$.

The next result gives conditions for the existence of compact attractors \mathcal{A}_h for $T_h(t)$ and the upper semicontinuity of these sets “at $h = 0$ ”.

THEOREM 2.4. *Assume that $T(t)$ has a local, compact attractor \mathcal{A} and that the hypotheses of Proposition 2.2 are satisfied. If each $T_h(t)$ is asymptotically smooth, then there is $h_0 > 0$ such that, for $0 < h \leq h_0$, $T_h(t)$ admits a local, compact attractor \mathcal{A}_h , which attracts $N_1 \cap X_h$. Moreover, $\delta_X(\mathcal{A}_h, \mathcal{A}) \rightarrow 0$ as $h \rightarrow 0$.*

Proof. From Proposition 2.2, it follows that $T_h(t)(N_1 \cap X_h)$, $t \geq 0$, belongs to a bounded set in X_h . The results in Hale, LaSalle and Slemrod [13] (see also [12]) imply the existence of a compact attractor \mathcal{A}_h for $T_h(t)$ which attracts $N_1 \cap X_h$. Owing to Relation (2.4), we can take $\mathcal{A}_h \subset \mathcal{N}(\mathcal{A}, \varepsilon_0)$. Since ε_0 is arbitrary, we obtain the result. \square

COROLLARY 2.5. *Assume that $T(t)$ has a local compact attractor \mathcal{A} and that the conditions of Proposition 2.2 are satisfied. If each space X_h is finite-dimensional, the conclusions of Theorem 2.4 hold.*

In the general case, the hypotheses of Theorem 2.4 do not enable us to give any information about the distance $\delta_X(\mathcal{A}, \mathcal{A}_h)$. However, if \mathcal{A} is reduced to a point x_0 , then, of course, under the hypotheses of Theorem 2.4, $\delta(\mathcal{A}, \mathcal{A}_h) \rightarrow 0$.

In the next result, we assume the attractors for the approximate semigroups exist and conclude that the original semigroup admits an attractor.

PROPOSITION 2.6. *Suppose there are bounded open sets $N_1 \subset N_2 \subset X$ and positive constants $\varepsilon_0, \bar{\varepsilon}_0, h_0, t_0, \delta_0$ such that, for each $0 < h \leq h_0$, the semigroup $T_h(t)$ has a local compact attractor $\mathcal{A}_h \subset X$, with $\mathcal{N}(\mathcal{A}_h, \bar{\varepsilon}_0) \subset N_1$, and that*

- (i) \mathcal{A}_h attracts N_1 uniformly, that is, for any $\varepsilon_1 > 0$, there is a $\tau_1 > 0$, independent of h , such that $T_h(t)(N_1 \cap X_h) \subset \mathcal{N}(\mathcal{A}_h, \varepsilon_1)$ for $t \geq \tau_1$,
- (ii) $T_h(t)(N_1 \cap X_h) \subset N_2 \cap X_h$, for all $t \geq 0$,
- (iii) $T(t)N_1 \subset N_2$ for $0 \leq t \leq t_0$,
- (iv) $T(t)x$ is well defined for $x \in \mathcal{N}(N_2, \varepsilon_0)$ for $0 \leq t \leq \delta_0$.

Also assume that $T_h(t)$ conditionally approximates $T(t)$ on $\mathcal{N}(N_2, \varepsilon_0)$ uniformly on compact sets of $[t_0, +\infty)$. Then, there exists $\tau \geq t_0$ such that, for $t \geq \tau$,

$$(2.5) \quad T(t)N_1 \subset N_1.$$

If, in addition, $T(t)$ is asymptotically smooth, then $T(t)$ has a local compact attractor \mathcal{A} attracting N_1 and, for any $\varepsilon > 0$, there exists $h_1 > 0$ such that, for $0 < h \leq h_1$,

$$(2.6) \quad \mathcal{A} \subset \mathcal{N}(\mathcal{A}_h, \varepsilon).$$

Proof. Let us first show that

$$(2.7) \quad T(t)N_1 \subset \mathcal{N}(N_2, \varepsilon_0) \quad \text{for all } t \geq 0.$$

Owing to (iii), $T(t)N_1 \subset \mathcal{N}(N_2, \varepsilon_0)$ for $0 \leq t \leq t_0$. Suppose that the property (2.7) is not true; then there exist $x \in N_1$ and $t_2 > t_0$ such that $T(t_2)x \in \partial(\mathcal{N}(N_2, \varepsilon_0))$ and $T(t)x \in \mathcal{N}(N_2, \varepsilon_0)$ for $0 \leq t < t_2$ (the existence of t_2 is ensured by (iv)). Thanks to the approximation property (2.1), there exist a positive number h_2 and, for $0 < h \leq h_2$, an element $x_h \in N_1 \cap X_h$ close enough to x so that, for $0 \leq t \leq t_2$, $0 < h \leq h_2$,

$$(2.8) \quad \|T(t)x_h - T(t)x\|_X < \varepsilon_0/3.$$

Moreover, there exists $h_3 > 0$, with $h_3 \leq \inf(h_0, h_2)$, such that, for $0 < h \leq h_3$,

$$(2.9) \quad \eta(h, [t_0, t_2], \mathcal{N}(N_2, \varepsilon_0)) < \varepsilon_0/3.$$

Thus, since $T_h(t)$ conditionally approximates $T(t)$ on $\mathcal{N}(N_2, \varepsilon_0)$ uniformly on $[t_0, t_2]$, (2.8), (2.9) and (ii) imply that $T(t_2)x \in \text{Cl}(\mathcal{N}(N_2, 2\varepsilon_0/3))$, which is a contradiction. Thus (2.7) is true.

Next we show that $T(t)N_1 \subset N_1$ for $t \geq \tau$, where $\tau \geq t_0$ is a constant. Owing to the property (i), there exists $\tau \geq t_0$ such that, for $0 < h \leq h_0$, $T_h(t)(N_1 \cap X_h) \subset \mathcal{N}(\mathcal{A}_h, \bar{\varepsilon}_0/4)$ for $t \geq \tau$. Now let $x \in N_1$ be given. As above, there exist a positive

number h_4 , with $h_4 \leq h_0$, and, for $0 < h \leq h_4$, an element $x_h \in N_1 \cap X_h$ close to x such that, for $0 < h \leq h_4$,

$$(2.10) \quad \|T(t)x_h - T(t)x\|_X < \bar{\varepsilon}_0/3 \quad \text{for all } t, \text{ with } 0 \leq t \leq 2\tau$$

and

$$(2.11) \quad \eta(h, [t_0, 2\tau], \mathcal{N}(N_2, \varepsilon_0)) < \bar{\varepsilon}_0/3.$$

As (2.7) holds and $T_h(t)$ conditionally approximates $T(t)$ on $\mathcal{N}(N_2, \varepsilon_0)$ uniformly on $[t_0, 2\tau]$, we derive from (2.10) and (2.11) that $T(t)x \in \mathcal{N}(\mathcal{A}_h, 3\bar{\varepsilon}_0/4) \subset N_1$, for $\tau \leq t \leq 2\tau$.

An easy induction, similar to the one of the proof of Lemma 2.1, shows that $T(t)x \in N_1$, for $t \geq \tau$.

If, moreover, $T(t)$ is asymptotically smooth, we conclude, by using a result of [7], that $T(t)$ has a compact attractor $\mathcal{A} \subset N_1$ attracting N_1 . It remains to prove (2.6). Let $\varepsilon > 0$ be given. By (i), there exists $\tau_1 > t_0$, independent of h , such that $T_h(t)(N_1 \cap X_h) \subset \mathcal{N}(\mathcal{A}_h, \varepsilon/3)$ for $t \geq \tau_1$ and for $0 < h \leq h_0$. Because of the compactness of \mathcal{A} , there exists h_5 , $0 < h_5 \leq h_0$, such that, for $0 < h \leq h_5$, with each element $x \in \mathcal{A}$ we can associate an element $P_h x$ in $N_1 \cap X_h$ such that

$$\|T(t)x - T(t)P_h x\|_X \leq \varepsilon/3 \quad \text{for } 0 \leq t \leq \tau_1.$$

Finally, there exists h_1 , $0 < h_1 \leq h_5$, such that, for any $x_h \in N_1 \cap X_h$,

$$\|T(t)x_h - T_h(t)x_h\|_X \leq \varepsilon/3 \quad \text{for } t_0 \leq t \leq \tau_1.$$

Thus, for $0 < h \leq h_1$, $T(\tau_1)x \in \mathcal{N}(\mathcal{A}_h, \varepsilon)$ for all x in \mathcal{A} ; and from the equality $T(\tau_1)\mathcal{A} = \mathcal{A}$ we deduce the inclusion (2.6). \square

Remark 2.7. Property (2.6) means that $\delta_X(\mathcal{A}, \mathcal{A}_h) \rightarrow 0$ as $h \rightarrow 0$. Let us remark that, under the hypotheses of Proposition 2.6, $\delta_X(\mathcal{A}_h, \mathcal{A})$ also tends to 0 as h tends to 0. Indeed, as \mathcal{A} attracts N_1 , for any $\varepsilon_1 > 0$, there exists $\tau_1 > t_0$ such that $T(t)\mathcal{A}_h \subset \mathcal{N}(\mathcal{A}, \varepsilon_1/2)$ for $t \geq \tau_1$. On the other hand, there exists $\bar{h} > 0$ such that, for $0 < h \leq \bar{h}$, $\eta(h, [t_0, \tau_1], \mathcal{N}(N_2, \varepsilon_0)) \leq \varepsilon_1/2$. Thus, $T_h(\tau_1)\mathcal{A}_h \subset \mathcal{N}(\mathcal{A}, \varepsilon_1)$ and, since $T_h(\tau_1)\mathcal{A}_h = \mathcal{A}_h$, $\mathcal{A}_h \subset \mathcal{N}(\mathcal{A}, \varepsilon_1)$ for $0 < h \leq \bar{h}$.

Remark 2.8. The assumption (i) in Proposition 2.6 that \mathcal{A}_h attracts N_1 uniformly is a very strong condition. However, one would expect numerical procedures to have such a property. The detailed structure of the flow on the attractor \mathcal{A}_h could vary considerably with h . This depends on the flow defined by $T(t)$. Consider, for example, a scalar equation $\dot{u} = f(u)$, $u \in R$, where the flow is given by $\rightarrow \bullet \rightarrow \bullet \leftarrow$. If one approximates this flow numerically, two situations could arise. One could obtain either the approximate flow $\rightarrow \bullet \leftarrow$ or $\rightarrow \bullet \leftarrow \bullet \rightarrow \bullet \leftarrow$. The global attractor in one case is a point and in the other is a line segment. The global attractor for the original problem is a line segment. For one of the approximation schemes, the attractors \mathcal{A}_h approach a point as $h \rightarrow 0$ which is a local attractor for $T(t)$ and, for the other, \mathcal{A}_h approaches a line segment which is the global attractor for $T(t)$. If the flow on the attractor for $T(t)$ is less sensitive to small perturbations, this situation will not arise.

Let us now turn to the question of how close \mathcal{A}_h is to \mathcal{A} with the measure of closeness given by $\delta_X(\mathcal{A}_h, \mathcal{A})$. We give some results in this direction for some particular cases.

PROPOSITION 2.9. *Suppose the hypotheses of Theorem 2.4 are satisfied with the associated function $\eta(h, I, N_3) = ch^{\gamma_0}$ for some positive constants c, γ_0 , independent of h and $I \subset [t_0, \infty)$. Then there is a constant $c_1 > 0$ such that $\delta_X(\mathcal{A}_h, \mathcal{A}) \leq c_1 h^{\gamma_0}$ for $0 < h \leq h_0$.*

Proof. The proof follows from the proof of Proposition 2.2 and Theorem 2.4 using the special function $\eta(h, I, N_3) = ch^{\gamma_0}$. \square

The hypothesis on $\eta(h, I, N_3)$ in Proposition 2.9 is not usually satisfied. A more reasonable condition on $\eta(h, I, N_3)$ is given in the next result, but then we must impose stronger attractivity properties of \mathcal{A} .

PROPOSITION 2.10. *Assume the hypotheses of Theorem 2.4 are satisfied with the associated function $\eta(h, [t_0, t_1], N_3) = c_0 h^{\gamma_0} e^{\alpha_0 t_1}$ for some positive constants c_0, γ_0, α_0 , independent of h and t_1 . If there are an open neighborhood U of \mathcal{A} and positive constants c_1, β_0 such that*

$$\delta_X(T(t)U, \mathcal{A}) \leq c_1 e^{-\beta_0 t}, \quad t \geq 0,$$

then, for $h \leq h_0$, we have

$$\delta_X(\mathcal{A}_h, \mathcal{A}) \leq c h^{\gamma_0 \beta_0 / (\alpha_0 + \beta_0)}$$

for some positive constant c .

Proof. If

$$t_1 = -\frac{1}{\beta_0} \log \frac{c_0}{c_1} h^{\gamma_0 \beta_0 / (\alpha_0 + \beta_0)},$$

then $\delta_X(T(t)U, \mathcal{A}) \leq c_0 h^{\gamma_0 \beta_0 / (\alpha_0 + \beta_0)}$ for $t \geq t_1$. Since \mathcal{A}_h is invariant, for any $x_h \in \mathcal{A}_h$, there is a $y_h \in \mathcal{A}_h$ such that $x_h = T_h(t_1)y_h$. If $x = T(t_1)y_h$, then

$$\|x_h - x\|_X = \|T_h(t_1)y_h - T(t_1)y_h\|_X \leq c_0^{1-\alpha_0/\beta_0} c_1 h^{\gamma_0 \beta_0 / (\alpha_0 + \beta_0)}.$$

This completes the proof. \square

Remark 2.11. If $T(t)$ is a gradient system (for the definition, see Hale [12]) for which there is a $t_1 > 0$ such that $T(t)$ is either compact for $t > t_1$ or an α -contraction, and if the set of equilibrium points E (i.e., the points ϕ such that $T(t)\phi = \phi, t \geq 0$) is bounded, then we know that $T(t)$ has a compact attractor \mathcal{A} . If, in addition, each element of E is hyperbolic, then E is a finite set, $\dim W^u(\phi) < +\infty$ and $\mathcal{A} = \bigcup_{\phi \in E} W^u(\phi)$, where $W^u(\phi)$ is the unstable set of ϕ . Furthermore, if the stable and unstable manifolds intersect transversally, there is an open neighborhood U of \mathcal{A} such that $\delta_X(T(t)U, \mathcal{A}) \rightarrow 0$ exponentially as $t \rightarrow +\infty$.

Thus, if the approximate semigroups $T_h(t)$ satisfy the hypothesis of Theorem 2.4 with $\eta(h, [t_0, t_1], N_3) = c_0 h^{\gamma_0} e^{\alpha_0 t_1}$, $T_h(t)$ admits a local compact attractor \mathcal{A}_h for h small enough and, by Proposition 2.10, we obtain a good estimate of $\delta_X(\mathcal{A}_h, \mathcal{A})$.

Now assume that, for $h > 0$, $T_h(t)$ is a gradient system. Then, one can prove that, for h small enough, the set of equilibrium points E_h of T_h is finite and has the same cardinality as E , and one can give an estimate of $\delta_X(E, E_h)$ and $\delta_X(E_h, E)$. Moreover, $\mathcal{A}_h = \bigcup_{\phi_h \in E_h} W^u(\phi_h)$, where $W^u(\phi_h)$ is the unstable set of ϕ_h . (For more details, see Lin and Raugel [25]).

In Remark 2.11 we have encountered a situation where the conditions of Proposition 2.10 are satisfied. One would expect that the hypothesis in Proposition 2.10

that $T(t)U \rightarrow \mathcal{A}$ exponentially as $t \rightarrow +\infty$ will be satisfied in specific evolutionary problems, at least generically with respect to the vector fields. A more precise statement is needed and certainly is nontrivial.

Let us end this section by pointing out that in some cases the semigroups $T_h(t)$ do not conditionally approximate $T(t)$ on any open set $U \subset X$. In this case, one has to use other ways to prove that $T_h(t)$ admits a local compact attractor \mathcal{A}_h for h small enough. In Subsection 6.2 we shall encounter a typical example of this case.

3. Galerkin Approximation of Sectorial Evolutionary Equations.

3.1. Galerkin Approximation of a Parabolic Equation. Let V and H be two (real) Hilbert spaces such that V is included in H with a continuous and dense imbedding; the space H is identified with its dual space, and the inner product of H , as well as the duality pairing between V and its dual space V' , is denoted by (\cdot, \cdot) (so we have the inclusions $V \subset H \subset V'$, where the imbeddings are continuous and dense). We introduce a continuous, bilinear form on $V \times V: (u, v) \in V \times V \rightarrow a(u, v)$ and the corresponding operator $A \in \mathcal{L}(V; V')$ defined by

$$\forall u, v \in V, \quad a(u, v) = (Au, v).$$

We denote by C_0 the constant of continuity of the bilinear form $a(\cdot, \cdot)$. We also suppose that there are two constants $\gamma > 0$ and $\gamma_0 \geq 0$ such that

$$(3.1) \quad \forall v \in V, \quad a(v, v) + \gamma_0 \|v\|_H^2 \geq \gamma \|v\|_V^2.$$

Moreover, if

$$b(u, v) = a(u, v) - a(v, u),$$

we assume that there exists a constant $C_1 > 0$ such that

$$(3.2) \quad |b(u, v)| \leq C_1 \|u\|_V \|v\|_H.$$

Now we consider the nonlinear equation

$$(3.3) \quad \begin{cases} du/dt + Au = f(u), \\ u(0) = u_0, \end{cases}$$

where u_0 belongs to V and $f: V \rightarrow H$ is locally Lipschitz continuous (i.e., f is continuous and, for any bounded set B of V , there is a constant k_B such that $\|f(u) - f(v)\|_H \leq k_B \|u - v\|_V$ for u, v in B).

Remark 3.1. We may always assume that $\gamma_0 = 0$. If $\gamma_0 > 0$, we can set $A_1 = A + \gamma_0 I$ and replace Eq. (3.3) by

$$(3.3)' \quad \begin{cases} du/dt + A_1 u = f(u) + \gamma_0 u, \\ u(0) = u_0. \end{cases}$$

Therefore, we assume in the sequel that $\gamma_0 = 0$.

Let $D(A) = \{v \in V; Av \in H\}$. Note that $D(A)$ is dense in V and H and that A is a sectorial operator on H in the sense of Henry [17], so that we can define the operators A^α for any $\alpha \geq 0$. (We recall that A is sectorial if and only if the semigroup e^{-At} generated by A is an analytic semigroup.) If we define $X^\alpha = D(A^\alpha)$, $\alpha \geq 0$, with the graph norm $\|v\|_{X^\alpha} = \|A^\alpha v\|_H$, $v \in X^\alpha$, then X^α is a Hilbert space normed by $\|v\|_{X^\alpha}$ and $X^0 = H$ (for more details, see Henry [17, pp. 26–29]). From the hypothesis (3.2) we derive at once that $D(A) = D(A^*)$, where

A^* is the adjoint operator of A , defined by $(A^*u, v) = a(v, u)$ for any u, v in V . By using a result of Lions ([26, Théorème 5.3]) we conclude that

$$(3.4) \quad D(A^{1/2}) = D(A^{*1/2}) = V.$$

Thus we are in the context considered in [17, Chapter 3], so that we can use existence, uniqueness and regularity results for the solution of the equation (3.3). In particular, we know that under the above hypotheses on A , f and u_0 there is a unique solution in V of Eq. (3.3) on a maximal interval of existence $(0, \tau_{u_0})$. Here we assume that all solutions are defined for $t \geq 0$, so that we can introduce the map $T(t): V \rightarrow V$, $t \geq 0$, defined by $T(t)u_0 = u(t, u_0)$ and obtain a C^0 -semigroup on V . We also suppose that $T(t)$ has a (local) compact attractor \mathcal{A} which attracts a bounded open set \mathcal{O} , $\mathcal{O} \supset \mathcal{A}$ (see [12] for the existence of \mathcal{A}).

Now let us turn to a finite-dimensional approximation of Eq. (3.3). Let $h > 0$ be a real parameter which will tend to 0 and $(V_h)_h$ a family of finite-dimensional subspaces of V . We introduce the operator $A_h \in \mathcal{L}(V_h; V_h)$ defined by

$$(3.5) \quad \forall v_h \in V_h, \quad (A_h w_h, v_h) = a(w_h, v_h) \quad \text{for } w_h \text{ in } V_h.$$

Let $Q_h \in \mathcal{L}(H; V_h)$ be the projector on V_h in the space H , i.e.,

$$\forall v \in H, \quad \forall v_h \in V_h, \quad (v - Q_h v, v_h) = 0,$$

and let $P_h \in \mathcal{L}(V; V_h)$ be the projector on V_h in the space V , i.e.,

$$\forall v \in V, \quad \forall v_h \in V_h, \quad a(v - P_h v, v_h) = 0.$$

Now consider the following equation in V_h :

$$(3.3)_h \quad \begin{cases} du_h/dt + A_h u_h = Q_h f(u_h), \\ u_h(0) = u_{0h}, \end{cases}$$

where $u_{0h} \in V_h$. Equation (3.3)_h is an ordinary differential equation. We introduce the map $T_h(t): V_h \rightarrow V_h$, defined by $T_h(t)u_{0h} = u_h(t, u_{0h})$ as long as $u_h(t, u_{0h})$ exists. $T_h(t)u_{0h}$ is continuous in t and u_{0h} when it is well defined and, if $T_h(t)u_{0h}$ exists on $[t_0, t_1]$, it is left-continuous at t_1 .

In order to prove that $T_h(t)$ also admits a compact attractor \mathcal{A}_h , for h small enough, we need the following additional hypotheses on the spaces $(V_h)_h$:

There exist an integer $m > 0$ and, for any β , $\frac{1}{2} \leq \beta \leq 1$, a constant $C(\beta) > 0$ such that, for all w in $X^\beta \equiv D(A^\beta)$,

$$(3.6)(i) \quad \|w - P_h w\|_V + \|w - Q_h w\|_V \leq C(\beta)h^{2m(\beta-1/2)} \|w\|_{X^\beta},$$

and

$$(3.6)(ii) \quad \|w - P_h w\|_H + \|w - Q_h w\|_H \leq C(\beta)h^{2m\beta} \|w\|_{X^\beta}.$$

Remark 3.2. The hypotheses (3.6) (i) and (3.6) (ii) are realistic and are satisfied in many cases when A is an elliptic differential operator (see Ciarlet [5] and Example 3.1 below). We point out that in this case the quantity $2m$ in (3.6) is rather related to the order of the differential operator A than to the order of the ‘‘approximation’’ of V by V_h . In Example 3.1 below, $2m$ remains equal to 2 (which is the order of \mathcal{L}), even if we replace $P_1(K)$ in (3.8) by a space of higher-order polynomials.

Example 3.1. Let Ω be a regular bounded domain or a convex bounded set in \mathbf{R}^2 . In Ω we are given an elliptic operator of the following form:

$$(3.7) \quad \mathcal{L}v = \sum_{i,j=1}^2 a_{ij}(x) \frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{j=1}^2 b_j(x) \frac{\partial v}{\partial x_j} + c(x)v,$$

where the coefficients a_{ij}, b_j, c are smooth enough and where \mathcal{L} is assumed to be uniformly and strongly elliptic. If A denotes the operator $-\mathcal{L}$, with homogeneous Dirichlet boundary conditions, then the hypotheses (3.1) and (3.4) are satisfied with $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$, $D(A^{1/2}) = V = H_0^1(\Omega)$, $H = L^2(\Omega)$. And one can find finite-dimensional subspaces V_h of $H_0^1(\Omega)$ such that the conditions (3.6) (i), (3.6) (ii) are satisfied with $m = 1$. For instance, if Ω is a convex polygonal domain, we introduce a uniformly regular family $(\mathcal{T}_h)_h$ of triangulations in the sense of [5], where \mathcal{T}_h is made of triangles with diameters bounded by h . And we set

$$(3.8) \quad V_h = \{v_h \in C^0(\overline{\Omega}) \cap H_0^1(\Omega) : \forall K \in \mathcal{T}_h, v_h|_K \in P_1(K)\},$$

where $P_1(K)$ is the space of all polynomials of degree ≤ 1 on K . In this case, the hypotheses (3.6) (i) and (3.6) (ii) are satisfied with $m = 1$. Moreover, even if the family $(\mathcal{T}_h)_h$ is only regular, the hypothesis (3.6) (ii) is satisfied and the condition (3.6) (i) usually holds (see Crouzeix and Thomée [9]).

THEOREM 3.1. *Under the above hypotheses, there exists $h_0 > 0$ such that, for $h \leq h_0$, $T_h(t)$ admits a local compact attractor \mathcal{A}_h , which attracts an open set $N_1 \cap V_h$, where N_1 is independent of h . Moreover, $\delta_V(\mathcal{A}_h, \mathcal{A}) \rightarrow 0$ as $h \rightarrow 0$.*

Proof. We shall prove that the hypotheses of Proposition 2.2 are satisfied by $T(t)$ and $T_h(t)$ for h small enough. Clearly, it is sufficient to show that there are constants $h_0 > 0$, $\delta_0 > 0$ and $t_0 > 0$ and three open neighborhoods N_1, N_2, N_3 of \mathcal{A} with $N_1 \subset \mathcal{O}$, $N_1 \subset N_2 \subset \mathcal{N}(N_2, \delta_0) \subset N_3$, such that the conditions (i) and (ii) of Proposition 2.2 are satisfied and that $T_h(t)$ conditionally approximates $T(t)$ on N_3 uniformly on compact sets of $[t_0, +\infty)$. Let us prove this in two steps.

First step. As \mathcal{A} is a compact attractor, there is a bounded open neighborhood N_1 of \mathcal{A} such that $N_1 \subset \mathcal{O}$ and $T(t)N_1 \subset N_1$ for $t \geq 0$. We choose a real number $\varepsilon_0 \geq 8B_0C_0/\gamma$, where $B_0 = \max_{v \in N_1} \|v\|_V$, and we set $N_2 = \mathcal{N}(N_1, \varepsilon_0)$. Finally, let δ_0 be a positive real number and define $N_3 = \mathcal{N}(N_2, \delta_0)$. Now we want to prove that there exists a constant $t_0 > 0$ such that $T_h(t)(N_1 \cap V_h) \subset N_2$ for $0 \leq t \leq t_0$. Using classical arguments of the theory of differential equations, we easily see that it is sufficient to prove the following property:

$$(A) \quad \left\{ \begin{array}{l} \text{There exists a constant } t_0 > 0 \text{ independent of } h \text{ such that, for} \\ \text{any } u_{0h} \in N_1 \cap V_h, \text{ if } T_h(t)u_{0h} \text{ belongs to } \mathcal{N}(u_{0h}, \varepsilon_0 + \delta_0) \text{ for} \\ 0 \leq t \leq t(u_{0h}), \text{ where } t(u_{0h}) \leq t_0, \text{ then } T_h(t)u_{0h} \in \mathcal{N}(u_{0h}, \varepsilon_0) \\ \text{for } 0 \leq t \leq t(u_{0h}). \end{array} \right.$$

As f is globally Lipschitz continuous on $\mathcal{N}(N_3, \delta_0)$, there exist constants $M_1 > 0$ and $L > 0$ such that

$$(3.9)(i) \quad \forall v \in \mathcal{N}(N_3, \delta_0), \quad \|f(v)\|_H \leq M_1,$$

and

$$(3.9)(ii) \quad \forall v, w \in \mathcal{N}(N_3, \delta_0), \quad \|f(v) - f(w)\|_H \leq L\|v - w\|_V.$$

If u_h is the solution of Eq. (3.3) $_h$, $u_h - u_{0h}$ satisfies the equation

$$(3.10) \quad d(u_h - u_{0h})/dt + A_h(u_h - u_{0h}) = Q_h f(u_h) + A_h u_{0h}.$$

Taking the inner product in H of the equation (3.10) by $d(u_h - u_{0h})/dt$, we obtain

$$(3.11) \quad \begin{aligned} & \left\| \frac{d}{dt}(u_h - u_{0h}) \right\|_H^2 + a \left(u_h - u_{0h}, \frac{d}{dt}(u_h - u_{0h}) \right) \\ &= \left(f(u_h), \frac{d}{dt}(u_h - u_{0h}) \right) + \frac{d}{dt} a(u_{0h}, u_h - u_{0h}). \end{aligned}$$

But

$$(3.12) \quad \begin{aligned} a \left(u_h - u_{0h}, \frac{d}{dt}(u_h - u_{0h}) \right) &= \frac{1}{2} \frac{d}{dt} a(u_h - u_{0h}, u_h - u_{0h}) \\ &+ \frac{1}{2} b \left(u_h - u_{0h}, \frac{d}{dt}(u_h - u_{0h}) \right), \end{aligned}$$

so that we deduce from (3.11) and (3.12), by using the inequality (3.2), that

$$\begin{aligned} & \left\| \frac{d}{dt}(u_h - u_{0h}) \right\|_H^2 + \frac{1}{2} \frac{d}{dt} a(u_h - u_{0h}, u_h - u_{0h}) \\ & \leq M_1 \left\| \frac{d}{dt}(u_h - u_{0h}) \right\|_H + C_1 \|u_h - u_{0h}\|_V \left\| \frac{d}{dt}(u_h - u_{0h}) \right\|_H \\ & \quad + \frac{d}{dt} a(u_{0h}, u_h - u_{0h}), \end{aligned}$$

which implies that

$$(3.13) \quad \frac{d}{dt} a(u_h - u_{0h}, u_h - u_{0h}) \leq M_1^2 + C_1^2 \|u_h - u_{0h}\|_V^2 + 2 \frac{d}{dt} a(u_{0h}, u_h - u_{0h}).$$

Finally, integrating (3.13) from 0 to t_h and using (3.1) (with $\gamma_0 = 0$) and the inequality $ab \leq a^2/2\varepsilon + \varepsilon b^2/2$, we obtain

$$\|u_h(t_h) - u_{0h}\|_V^2 \leq \frac{2t_h M_1^2}{\gamma} + \frac{2C_1^2}{\gamma} \int_0^{t_h} \|u_h(s) - u_{0h}\|_V^2 ds + \frac{4C_0^2}{\gamma^2} \|u_{0h}\|_V^2.$$

Thanks to Gronwall's inequality, we derive from the above estimate that

$$(3.14) \quad \|u_h(t_h) - u_{0h}\|_V^2 \leq \left(\frac{2t_h M_1^2}{\gamma} + \frac{4C_0^2}{\gamma^2} \|u_{0h}\|_V^2 \right) e^{(2C_1^2/\gamma)t_h}.$$

If $u_{0h} \in N_1 \cap V_h$, (3.14) becomes

$$(3.15) \quad \|u_h(t_h) - u_{0h}\|_V^2 \leq \left(\frac{2t_h M_1^2}{\gamma} + \frac{4C_0^2}{\gamma^2} B_0^2 \right) e^{(2C_1^2/\gamma)t_h}.$$

From (3.15) it is clear that there exists a constant $t_0 > 0$ independent of h such that property (A) holds.

It remains to prove that $T_h(t)$ conditionally approximates $T(t)$ on N_3 uniformly on compact sets of $[t_0, +\infty)$.

Step 2. Estimate of $\|T_h(t)u_{0h} - T(t)u_{0h}\|_V$ for $t_0 \leq t \leq t_1$ when $T_h(\tau)u_{0h}$ and $T(\tau)u_{0h}$ belong to N_3 for $0 \leq \tau \leq t$. To this end, we estimate the term $\|\tau T_h(\tau)u_{0h} - \tau T(\tau)u_{0h}\|_V$ for $0 \leq \tau \leq t$. We set $u(\tau) = T(\tau)u_{0h}$, $u_h(\tau) = T_h(\tau)u_{0h}$,

$Z(\tau) = \tau u(\tau)$ and $Z_h(\tau) = \tau u_h(\tau)$. Using the equality $Q_h A = A_h P_h$, one easily shows that $Z_h(\tau) - P_h Z(\tau)$ satisfies the equation

$$(3.16) \quad \begin{aligned} & \frac{d}{dt}(Z_h - P_h Z) + A_h(Z_h - P_h Z) \\ &= \tau Q_h(f(u_h) - f(u)) + \tau \left(\frac{d}{dt}(Q_h u - P_h u) \right) + u_h - P_h u. \end{aligned}$$

Taking the inner product in H of (3.16) by $d(Z_h - P_h Z)/dt$, we obtain, thanks to the property (3.9)(ii),

$$(3.17) \quad \begin{aligned} & \left\| \frac{d}{dt}(Z_h - P_h Z) \right\|_H^2 + a \left(Z_h - P_h Z, \frac{d}{dt}(Z_h - P_h Z) \right) \\ & \leq L \|Z_h - Z\|_V \left\| \frac{d}{dt}(Z_h - P_h Z) \right\|_H \\ & \quad + \left\| \frac{d}{dt}(Z - P_h Z) \right\|_H \left\| \frac{d}{dt}(Z_h - P_h Z) \right\|_H \\ & \quad + \|u_h - P_h u\|_H \left\| \frac{d}{dt}(Z_h - P_h Z) \right\|_H. \end{aligned}$$

Using the relation (3.12) (where $u_h - u_{0h}$ is replaced by $Z_h - P_h Z$) and the hypothesis (3.2) as well as the inequality $ab \leq a^2/2\varepsilon + \varepsilon b^2/2$, we derive from (3.17),

$$(3.18) \quad \begin{aligned} & \frac{d}{dt} a(Z_h - P_h Z, Z_h - P_h Z) \\ & \leq (L^2 + C_1^2) \|Z_h - P_h Z\|_V^2 + L^2 \|Z - P_h Z\|_V^2 \\ & \quad + \left\| \frac{d}{dt}(Z - P_h Z) \right\|_H^2 + \|u_h - P_h u\|_H^2. \end{aligned}$$

If we integrate (3.18) from 0 to t and then apply Gronwall's inequality, we obtain

$$(3.19) \quad \begin{aligned} & \|(Z_h - P_h Z)(t)\|_V^2 \\ & \leq C_2 e^{C_3 t} \left[\int_0^t \left\{ \|Z(s) - P_h Z(s)\|_V^2 + \|u_h(s) - Q_h u(s)\|_H^2 \right. \right. \\ & \quad \left. \left. + \|u(s) - Q_h u(s)\|_H^2 + \|u(s) - P_h u(s)\|_H^2 \right. \right. \\ & \quad \left. \left. + \left\| \frac{d}{dt}(Z(s) - P_h Z(s)) \right\|_H^2 \right\} ds \right], \end{aligned}$$

where C_2 and C_3 are two positive constants independent of h and t . We now estimate $\|u_h(\tau) - Q_h u(\tau)\|_H, 0 \leq \tau \leq t$. The function $Q_h u - u_h$ satisfies the equation

$$(3.20) \quad \begin{cases} \frac{d}{dt}(Q_h u - u_h) + A_h(Q_h u - u_h) \\ \quad = Q_h f(u) - Q_h f(u_h) + (A_h Q_h - Q_h A)u, \\ (Q_h u - u_h)(0) = 0. \end{cases}$$

Taking the inner product in H of (3.20) by $Q_h u - u_h$ and using the property (3.9)(ii), we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|u_h - Q_h u\|_H^2 + \gamma \|u_h - Q_h u\|_V^2 \\ & \leq L \|u - Q_h u\|_V \|u_h - Q_h u\|_H + L \|u_h - Q_h u\|_V \|u_h - Q_h u\|_H \\ & \quad + C_0 \|u - Q_h u\|_V \|u_h - Q_h u\|_V. \end{aligned}$$

By Gronwall's inequality, the above estimate implies, after an integration from 0 to τ ,

$$(3.21) \quad \|u_h(\tau) - Q_h u(\tau)\|_H^2 \leq C_4 e^{C_5 \tau} \left(\int_0^\tau \|u(s) - Q_h u(s)\|_V^2 ds \right),$$

where C_4 and C_5 are two positive constants independent of τ and h . Thanks to the properties (3.6)(i) and (3.6)(ii), we deduce from (3.19) and (3.21), for $0 \leq t \leq t_1$,

$$(3.22) \quad \begin{aligned} & \|Z(t) - Z_h(t)\|_V^2 \\ & \leq C_6 e^{C_7 t} h^{2m} \left\{ \|tAu(t)\|_H^2 + \int_0^t \left(\|Au(s)\|_H^2 + \left\| s \frac{du}{dt}(s) \right\|_V^2 \right. \right. \\ & \quad \left. \left. + \|u(s)\|_V^2 \right) ds \right\}, \end{aligned}$$

where C_6 and C_7 are two positive constants independent of t and h . On the one hand, using Henry [17, p. 71], one easily proves that, for $0 < \tau \leq t$,

$$(3.23) \quad \left\| \frac{du}{dt}(\tau) \right\|_V \leq K_0(N_2, \delta_0) e^{K_1 t_1} \frac{1}{\tau},$$

where $K_0(N_2, \delta_0)$ is a positive constant depending only on N_2 and δ_0 , and K_1 is a positive constant.

Since $\tau Au = \tau f(u) - \tau du/dt$, we infer from (3.23), for $0 < \tau \leq t$,

$$(3.24) \quad \|\tau Au(\tau)\|_H \leq t_1 \sup_{v \in N_3} \|f(v)\|_H + K_0(N_2, \delta_0) e^{K_1 t_1}.$$

On the other hand, since $du(\tau)/dt$ belongs to H for $\tau > 0$, we may consider the inner product in H of Eq. (3.3) by du/dt ; thus, using a relation similar to (3.12), we obtain, for $0 \leq t \leq t_1$,

$$(3.25) \quad \int_0^t \left\| \frac{du}{dt}(s) \right\|_H^2 ds \leq 2 \int_0^t \|f(u(s))\|_H^2 ds + 2C_1^2 \int_0^t \|u(s)\|_V^2 ds + C_0 \|u(t)\|_V^2.$$

Since $Au = f(u) - du/dt$, we deduce from (3.25) that, for $0 \leq t \leq t_1$,

$$(3.26) \quad \int_0^t \|Au(s)\|_H^2 ds \leq K_2(N_2, \delta_0) e^{K_3 t_1},$$

where $K_2(N_2, \delta_0)$ is a positive constant depending only on N_2 and δ_0 , and K_3 is a positive constant.

Finally, the estimates (3.22), (3.23), (3.24) and (3.26) give us, for $0 \leq t \leq t_1$,

$$\|Z(t) - Z_h(t)\|_V \leq K_4(N_2, \delta_0) e^{K_5 t_1} h^m$$

or, for $t_0 \leq t \leq t_1$,

$$(3.27) \quad \|u(t) - u_h(t)\|_V \leq K_4(N_2, \delta_0) \frac{e^{K_5 t_1} h^m}{t_0},$$

where $K_4(N_2, \delta_0)$ is a positive constant depending only on N_2 and δ_0 , and K_5 is a positive constant. \square

Remark 3.3. We also could have used the methods of Bramble, Schatz, Thomée and Wahlbin [3], Fujita and Mizutani [10], Helfrich [16] or Johnson, Larsson, Thomée and Wahlbin [20] for estimating $\|u(t) - u_h(t)\|_V$. For the estimate of $\|u(t) - u_h(t)\|_V$ when u is more regular, we refer the reader to Thomée [33] and to Thomée and Wahlbin [34]. (See also the references in [33]).

Remark 3.4. Let Ω be a regular or convex, bounded domain in \mathbf{R}^n , $n = 1, 2, 3$, and let $f: \mathbf{R} \rightarrow \mathbf{R}$ be a locally Lipschitz continuous function. Then, if $n = 1$, the mapping $f: u \in H^1(\Omega) \rightarrow f(u(x)) \in L^2(\Omega)$ is also locally Lipschitz continuous. If, in the cases $n = 2$ or 3 , f satisfies the additional condition

$$(3.28) \quad \forall v, \forall w \in \mathbf{R}, \quad |f(v) - f(w)| \leq C(1 + |v| + |w|)^\sigma |v - w|,$$

where

$$\sigma \leq \frac{2}{n-2} \quad \text{for } n \geq 3, \quad \sigma \text{ arbitrary for } n = 2,$$

then the mapping $f: u \in H^1(\Omega) \rightarrow f(u) \in L^2(\Omega)$ is also locally Lipschitz continuous. If the condition (3.28) is not satisfied, we have in general to work in a space other than $H^1(\Omega)$ (see Subsection 3.2 and Example 3.2 below).

3.2. An Extension of the Previous Result. Let us again consider the operator A introduced in Subsection 3.1 that satisfies the properties (3.1) and (3.2). Now we assume that $f: V \rightarrow H$ is no longer Lipschitz continuous. But instead, we suppose that A is a sectorial operator on a Banach space $Y \subset H$ and that $f: Y^\alpha \rightarrow Y$ is locally Lipschitz continuous, for a real number α , $\frac{1}{2} \leq \alpha < 1$. Furthermore, we assume that the following continuous inclusions hold:

$$(3.29) \quad D_Y(A) \hookrightarrow Y^\alpha \hookrightarrow V \hookrightarrow Y \hookrightarrow H,$$

where $D_Y(A) = \{y \in Y: Ay \in Y\}$ and $Y^\alpha = D_Y(A^\alpha)$.

We assume that all the solutions $u(t, u_0)$ of (3.3) are defined and belong to Y^α for $t \geq 0$, if $u_0 \in Y^\alpha$. Thus, the map $T_Y(t): Y^\alpha \rightarrow Y^\alpha$, $t \geq 0$, defined by $T_Y(t)u_0 = u(t, u_0)$, becomes a C^0 -semigroup on Y^α . Finally we suppose that $T_Y(t)$ admits a compact attractor \mathcal{A} which attracts a bounded open set $\mathcal{O} \supset \mathcal{A}$. Then there exists an open neighborhood \tilde{N}_1 of \mathcal{A} such that $\tilde{N}_1 \subset \mathcal{O}$ and $T_Y(t)\tilde{N}_1 \subset \tilde{N}_1$ for $t \geq 0$.

Now we introduce a function \tilde{f} which is *globally* Lipschitz continuous from V into H and coincides with f on \mathcal{O} . We consider the equation

$$(3.30) \quad \begin{cases} d\tilde{u}/dt + A\tilde{u} = \tilde{f}(\tilde{u}), \\ \tilde{u}(0) = u_0. \end{cases}$$

Obviously, if $u_0 \in \tilde{N}_1$ then $\tilde{u}(t, u_0) = u(t, u_0)$ for $t \geq 0$. Let $(V_h)_h$ be the family of finite-dimensional subspaces of V introduced in Subsection 3.1. We suppose that the spaces V_h are included in Y^α , satisfy the conditions (3.6) and the two following assumptions:

$$(3.31)(i) \quad \begin{aligned} &\text{for any } \beta, \alpha < \beta \leq 1, \text{ there exists a constant } \theta(\alpha, \beta) > 0 \text{ such} \\ &\text{that, for } v \text{ in } Y^\beta, \end{aligned}$$

$$\|v - P_h v\|_{Y^\alpha} \leq Ch^{2m\theta(\alpha, \beta)} \|v\|_{Y^\beta},$$

and

there exists a constant θ_α , $0 < \theta_\alpha < \frac{1}{2}$, such that, for any v_h in (3.31)(ii) V_h ,

$$\|v_h\|_{Y^\alpha} \leq Ch^{-2m\theta_\alpha} \|v_h\|_V.$$

We point out that, as in (3.6), the quantity $2m$ essentially depends on the order of the differential operator A . We consider the approximate problem

$$(3.30)_h \quad \begin{cases} d\tilde{u}_h/dt + A_h\tilde{u}_h = Q_h\tilde{f}(\tilde{u}_h), \\ \tilde{u}_h(0) = u_{0h}, \end{cases}$$

for $u_{0h} \in V_h$. We introduce the map $\tilde{T}_h(t): V_h \rightarrow V_h$ given by $\tilde{T}_h(t)u_{0h} = \tilde{u}_h(t, u_{0h})$. Since \tilde{f} is globally Lipschitz continuous, $\tilde{T}_h(t)u_{0h}$ exists for any $t \geq 0$.

THEOREM 3.2. *Under the above hypotheses, there exists $h_0 > 0$ such that, for $h \leq h_0$, $\tilde{T}_h(t)$ admits a compact attractor $\tilde{\mathcal{A}}_h$ which attracts the open set $\tilde{N}_1 \cap V_h$ (where \tilde{N}_1 is given above). Moreover, $\delta_{Y^\alpha}(\tilde{\mathcal{A}}_h, \mathcal{A}) \rightarrow 0$ as $h \rightarrow 0$.*

Proof. Let $t_0 > 0$ be a fixed real number. For any $t_1 > t_0$, we are going to estimate $\|T_Y(t)u_{0h} - \tilde{T}_h(t)u_{0h}\|_{Y^\alpha}$ for $t_0 \leq t \leq t_1$, when $u_{0h} \in \tilde{N}_1$. We set $u(t) = T_Y(t)u_{0h}$, $\tilde{u}_h(t) = \tilde{T}_h(t)u_{0h}$. Recall that $u(t) = \tilde{u}(t)$. By virtue of the conditions (3.31), we have

$$\begin{aligned} \|u(t) - \tilde{u}_h(t)\|_{Y^\alpha} &\leq \|\tilde{u}(t) - P_h\tilde{u}(t)\|_{Y^\alpha} + \|P_h\tilde{u}(t) - \tilde{u}_h(t)\|_{Y^\alpha} \\ &\leq Ch^{2m\theta(\alpha,\beta)} \|u(t)\|_{Y^\beta} + Ch^{-2m\theta_\alpha} \|P_h\tilde{u}(t) - \tilde{u}_h(t)\|_V, \end{aligned}$$

where $\alpha < \beta < 1$.

Arguing as in Subsection 3.1 (see estimate (3.27)), we obtain

$$(3.32) \quad h^{-2m\theta_\alpha} \|P_h\tilde{u}(t) - \tilde{u}_h(t)\|_V \leq K_4(\tilde{N}_1) \frac{e^{K_5 t_1}}{t_0} h^{2m(1/2-\theta_\alpha)}.$$

Finally, by using Henry [17, p. 57], we deduce from the above estimates, for $t_0 \leq t \leq t_1$,

$$(3.33) \quad \|u(t) - \tilde{u}_h(t)\|_{Y^\alpha} \leq K_6(\beta, \tilde{N}_1) \frac{e^{K_7 t_1}}{t_0} \sup(h^{2m(1/2-\theta_\alpha)}, h^{2m\theta(\alpha,\beta)}).$$

Since $\theta_\alpha < \frac{1}{2}$, (3.33) implies that the hypotheses of Proposition 2.1 hold and Theorem 3.2 is proved. \square

Example 3.2. Consider the equation

$$(3.34) \quad \begin{cases} du/dt - \Delta u = f(u), \\ u|_{\partial\Omega} = 0, \\ u(t)|_{t=0} = u_0, \end{cases}$$

where, for instance, Ω is a convex polygonal domain in \mathbf{R}^2 . If the function $f: \mathbf{R} \rightarrow \mathbf{R}$ is locally Lipschitz continuous, but does not satisfy the condition (3.28), we cannot work in the space $V = H_0^1(\Omega)$. The map $f: w \in Y^\alpha \rightarrow f(w) \in Y$ is locally Lipschitz continuous if $Y = L^2(\Omega)$ and $\alpha > \frac{1}{2}$, or, if $Y = L^p(\Omega)$, $p > 2$ and $\alpha \geq \frac{1}{2}$. (Indeed in both cases, $Y^\alpha \hookrightarrow L^\infty(\Omega)$.)

Now assume that (3.34) admits an attractor \mathcal{A} in Y^α which attracts a bounded set $\mathcal{O} \supset \mathcal{A}$. So we can introduce the quantity

$$(3.35) \quad B_1 = \max_{v \in \mathcal{O}} \|v\|_{L^\infty(\Omega)}.$$

One easily constructs a function \tilde{f} satisfying

$$(3.36) \quad \tilde{f}(x) = \begin{cases} f(x) & \text{for } |x| \leq B_1, \\ 0 & \text{for } |x| \geq 2B_1. \end{cases}$$

The map $\tilde{f}: w \in V \rightarrow \tilde{f}(w) \in H$ is globally Lipschitz continuous and coincides with f on \mathcal{O} .

Let us give an example of spaces V_h in the case $Y^\alpha = H^{2\alpha}(\Omega) \cap H_0^1(\Omega)$, $\frac{1}{2} < \alpha < 1$. Let $(\mathcal{T}_h)_h$ be a uniformly regular family of triangulations in the sense of [5]. We set

$$V_h = \{v_h \in C^1(\bar{\Omega}) \cap H_0^1(\Omega) : v_h|_K \in P_3(K), \forall K \in \mathcal{T}_h\},$$

where $P_3(K)$ is the space of all polynomials of degree ≤ 3 on K . Then, of course, the hypotheses (3.6) are satisfied with $m = 1$. Conditions (3.31)(i) and (3.31)(ii) hold with $\theta(\alpha, \beta) = \beta - \alpha$ and $\theta_\alpha = \alpha - \frac{1}{2}$.

3.3. Approximation of Sectorial Evolutionary Equations with Special Projection Methods. More generally, let A be a sectorial linear operator on a Banach space X , and consider the nonlinear equation (3.3) where now we assume that there exists a real number $\alpha \in [0, 1)$ such that $f: X^\alpha \rightarrow X$ is locally Lipschitz continuous and $u_0 \in X^\alpha$. As in Subsection 3.1, we introduce the map $T(t): X^\alpha \rightarrow X^\alpha$, $t \geq 0$, defined by $T(t)u_0 = u(t, u_0)$ and suppose that $T(t)$ has a (local) compact attractor \mathcal{A} which attracts an open set $\mathcal{O} \supset \mathcal{A}$. We assume that $\text{Re } \sigma(A) > \gamma > 0$, where $\sigma(A)$ denotes the spectrum of A (if not, we replace A by $A_1 = A + aI$, $\text{Re } \sigma(A_1) > \gamma$ and replace Eq. (3.3) by Eq. (3.3)'). We also suppose that $\sigma(A)$ consists of isolated points λ_n only, with no accumulation in the finite part of \mathbf{C} , and that each λ_n is of finite order. We arrange the points λ_n in such a way that

$$\gamma < \text{Re } \lambda_1 \leq \text{Re } \lambda_2 \leq \cdots \leq \text{Re } \lambda_n \leq \text{Re } \lambda_{n+1} \leq \cdots,$$

where $\text{Re } \lambda_n \rightarrow +\infty$ as $n \rightarrow +\infty$.

We denote by Φ_n the generalized eigenspace corresponding to λ_n , by P_N the projection from X onto the space $[\Phi_1, \Phi_2, \dots, \Phi_N]$ and by Q_N the projection $I - P_N$. We assume that, for $0 \leq \beta < 1$, $\|P_N\|_{\mathcal{L}(X^\beta; X^\beta)}$ is bounded by a constant $K_\beta > 0$, uniformly with respect to N . By [17, p. 21], for any $\varepsilon > 0$, for any integer N , there exists a constant $K_{\varepsilon, N}$ such that

$$(3.37) \quad \|A^j e^{-At} Q_N\|_{\mathcal{L}(X, X)} \leq K_{\varepsilon, N} \frac{e^{-(\text{Re } \lambda_{N+1} - \varepsilon)t}}{t^j} \quad \text{for } j = 0, 1.$$

Below, we assume that, for $0 \leq \beta < 1$,

$$(3.38) \quad \lim_{N \rightarrow +\infty} \frac{K_{\varepsilon, N}}{(\text{Re } \lambda_{N+1} - \varepsilon)^\beta} = 0,$$

this condition being usually satisfied.

Now let us consider the following equation on $X_N = P_N X$:

$$(3.3)_N \quad \begin{cases} du_N/dt + Au_N = P_N f(u_N), \\ u_N(0) = u_{0N}, \end{cases}$$

where $u_{0N} \in X_N$. Equation (3.3)_N is an ordinary differential equation. Let us introduce the map $T_N(t): X_N \rightarrow X_N$, defined by $T_N(t)u_{0N} = u_N(t, u_{0N})$, as long as $u_N(t, u_{0N})$ exists.

THEOREM 3.3. *Under the above hypotheses, there exists a number $N_0 > 0$ such that, for $N \geq N_0$, $T_N(t)$ admits a local compact attractor \mathcal{A}_N which attracts an open set $\mathcal{O}_1 \cap X_N$, where \mathcal{O}_1 is independent of N . Moreover, $\delta_{X^\alpha}(\mathcal{A}_N, \mathcal{A}) \rightarrow 0$ as $N \rightarrow +\infty$.*

The proof of this theorem is very easy and is a consequence of Proposition 2.2 (the complete proof can be found in Hale, Lin and Raugel [15]).

4. Semidiscretization in Time of Some Parabolic Problems. We keep the same notations and the same assumptions as in Subsection 3.1, but here we moreover assume that the operator A is *selfadjoint* and has a *compact resolvent*. (The generalization of the following results to the case where A is not selfadjoint, but satisfies the condition (3.2), is left to the reader.) As in Subsection 3.1, we assume that $\gamma_0 = 0$, and we consider the nonlinear equation

$$(4.1) \quad \begin{cases} du/dt + Au = f(u), \\ u(0) = u^0, \end{cases}$$

where $u^0 \in V$ and $f \in C^2(V; H)$, for instance. The hypotheses on f can be weakened. Now let us turn to a semidiscretization in time of Eq. (4.1) by a one-step method. More precisely, let k be a positive time increment, let $t_n = nk, n \geq 0$, and define an approximation u_n of the solution u of (4.1) at time t_n by the recursion formula

$$(4.1)_k \quad \begin{cases} u_{n+1} = (1 - (1 - \theta)kA)(1 + \theta kA)^{-1}u_n + k(1 + \theta kA)^{-1}f(u_n), \\ u_0 = u^0, \end{cases}$$

where $\frac{1}{2} < \theta \leq 1$.

Remark 4.1. The results that we are going to prove below are also valid if we replace $f(u_n)$ in $(4.1)_k$ by $f(\theta u_{n+1} + (1 - \theta)u_n)$. But then the “linearized” scheme $(4.1)_k$ becomes a nonlinear one.

More generally, the following results are also true if we replace $(4.1)_k$ by a scheme that is strictly accurate of order 1 in the sense of Brenner, Crouzeix, Thomée [4] and is of the form

$$(4.2)_k \quad \begin{cases} u_{n+1} = r(kA)u_n + k \sum_{j=1}^m q_j(kA)f(u_n), \\ u_0 = u^0, \end{cases}$$

where r, q_1, \dots, q_m are rational functions of the variable z which are bounded, as well as $zq_j(z), 1 \leq j \leq m$, for $z \geq 0$, and where $|r(z)| < 1$, for $z \geq 0$, and $|r(\infty)| \neq 1$. The proof, in the case of the scheme $(4.2)_k$, uses the same arguments as below and the property that $r(z)$ can be written as $(1 - zs(z))/(1 + \sigma z)$, where σ is a suitable positive constant (for more details, see Raugel [29]).

Now we introduce the mapping $T_k \in \mathcal{L}(V, V)$ defined by $T_k u^0 = u_1$, where u_1 is given by the formula $(4.1)_k$. For any integer $n \geq 1$, $T_k^n u^0 = u_n$. We remark that T_k is well defined on the whole space V and that $T_k^n : \mathbf{N} \rightarrow C^0(V, V)$ is a discrete semigroup. Although Sections 1 and 2 deal with C^0 -semigroups $T(t) : \mathbf{R}^+ \rightarrow C^0(V; V)$ only, the definitions and the results contained there obviously extend to discrete semigroups. For instance, a set $B \subset V$ is said to attract a set $C \subset V$ under T_k if, for any $\varepsilon > 0$, there is an integer $n_0 = n_0(B, C, \varepsilon)$ such that $T_k^n C \subset \mathcal{N}(B, \varepsilon)$

for $n \geq n_0$ (the definitions of a local attractor and an attractor are unchanged; for more details, see Hale [11], for instance).

Here we suppose that the map $T(t): V \rightarrow V$, $t \geq 0$, defined by $T(t)u^0 = u(t)$, where $u(t)$ is the solution of (4.1), admits a local, compact attractor \mathcal{A} which attracts a bounded open set \mathcal{O} , $\mathcal{O} \supset \mathcal{A}$.

THEOREM 4.1. *Under the above hypotheses, there exists $k_0 > 0$, such that, for $k \leq k_0$, the process T_k admits a local, compact attractor \mathcal{A}_k , which attracts an open set N_1 , where N_1 is independent of k , $N_1 \supset \mathcal{A}_k$ for every k . Moreover, $\delta_V(\mathcal{A}_k, \mathcal{A}) \rightarrow 0$ as $k \rightarrow 0$.*

The remainder of this section will be devoted to the proof of Theorem 4.1. But, beforehand, let us recall the following discrete analogue of Gronwall's lemma, the proof of which is left to the reader.

LEMMA 4.2. *Let $(a_n)_n, (b_n)_n, (c_n)_n$ be three sequences of positive real numbers such that $(c_n)_n$ is monotonically increasing and*

$$(4.3) \quad a_n + b_n \leq c_n + \lambda \sum_{m=0}^{n-1} a_m \quad \text{for } n \geq 1 \text{ and } \lambda > 0,$$

with $a_0 + b_0 \leq c_0$. Then, these sequences also satisfy

$$(4.4) \quad a_n + b_n \leq c_n \exp(\lambda n) \quad \text{for } n \geq 0.$$

Only for the sake of simplicity, we consider that the space V is equipped with the norm

$$(4.5) \quad \forall v \in V, \quad \|v\|_V = (Av, v)^{1/2}.$$

Hence, the dual norm on V' is given by

$$\forall v' \in V', \quad \|v'\|_{V'} = (A^{-1}v', v')^{1/2}.$$

Proof of Theorem 4.1. In order to prove Theorem 4.1, we shall apply the following modified version of Theorem 2.4, the proof of which is left to the reader. Clearly, the conclusions of Theorem 2.4 and hence of Theorem 4.1 hold, if the following conditions are satisfied:

There exist four positive constants $k_0, \delta_0, \delta_1, \alpha_0$, with $\alpha_0 > k_0$, and two open neighborhoods $\mathcal{N}_1, \mathcal{N}_2$ of \mathcal{A} , with $\mathcal{N}_1 \subset \mathcal{N}_2$, such that, for $0 < k \leq k_0$,

(i) T_k is an asymptotically smooth map (this condition holds in particular if $T_k = T_{1k} + T_{2k}$, where T_{1k} is completely continuous and T_{2k} is a linear strict contraction);

(ii) $T(t)\mathcal{N}_1 \subset \mathcal{N}_2$ for $t \geq 0$,

(iii) $T_k^n \mathcal{N}_1 \subset \mathcal{N}_2$ for $0 \leq n \leq \alpha_0/k$,

(iv) $T_k \mathcal{N}(\mathcal{N}_2, \delta_0) \subset \mathcal{N}_3$, where $\mathcal{N}_3 = \mathcal{N}(\mathcal{N}_2, \delta_0 + \delta_1)$; and

(v) for any $\alpha_1 > \alpha_0$, there exist a constant $k_0(\alpha_1, \mathcal{N}_3)$ with $0 < k_0(\alpha_1, \mathcal{N}_3) < k_0$, and a function $\eta(k, \alpha_1, \mathcal{N}_3)$ defined for $0 < k \leq k_0(\alpha, \mathcal{N}_3)$ such that

$$(4.6) \quad \lim_{k \rightarrow \infty} \eta(k, \alpha_1, \mathcal{N}_3) = 0,$$

and, for any $0 < k \leq k_0(\alpha_1, \mathcal{N}_3)$, if $u^0 \in \mathcal{N}_3$ has the property that $T_k^n u^0$ and $T(nk)u^0$ belong to \mathcal{N}_3 for $0 \leq n \leq \alpha_2/k$ and $0 \leq nk \leq \alpha_2 + k_0$, respectively (where $\alpha_0 \leq \alpha_2 \leq \alpha_1$), then

$$(4.7) \quad \|T_k^n u^0 - T(nk)u^0\|_V \leq \eta(k, \alpha_1, \mathcal{N}_3) \quad \text{for } \frac{\alpha_0}{k} \leq n \leq \frac{\alpha_2}{k}.$$

Now we show in four steps that the above conditions are satisfied.

(1) By (4.1)_k, we can write, for any $u^0 \in V$,

$$\begin{aligned} T_k u^0 &= [(1 + \theta kA)^{-1} u^0 + k(1 + \theta kA)^{-1} f(u^0)] - (1 - \theta)kA(1 + \theta kA)^{-1} u^0 \\ &\equiv T_{1k} u^0 + T_{2k} u^0. \end{aligned}$$

Let B be a bounded set in H ; for any $v \in B$, we have $\|kA(1 + \theta kA)^{-1} v\|_H \leq \|v\|_H$. Hence, for any fixed positive k , $(1 + \theta kA)^{-1} B$ is a bounded set in $D(A)$. Since $D(A) \subset V$ is a compact embedding, this proves that T_{1k} is completely continuous. On the other hand, as A is an elliptic operator, T_{2k} , for $k > 0$, is a linear strict contraction as soon as $2\theta - 1 > 0$. Condition (i) is proved.

(2) As \mathcal{A} is a compact attractor, there is a bounded open neighborhood \mathcal{N}_1 of \mathcal{A} such that $\mathcal{N}_1 \subset \mathcal{O}$ and $T(t)\mathcal{N}_1 \subset \mathcal{N}_1$ for $t \geq 0$. Let $B_0 = \max_{v \in \mathcal{N}_1} \|v\|_V$ and $B_1 = \max_{v \in \mathcal{N}_1} \|f(v)\|_H$; we set $\varepsilon_0 = 4(B_0^2 + B_1^2)^{1/2}$ and $\mathcal{N}_2 = \mathcal{N}(\mathcal{N}_1, \varepsilon_0)$. Finally, we choose a real number $\delta_0 > 0$ and we set $\delta_1 = 2[(B_0 + \varepsilon_0 + \delta_0)^2 + B_2^2]^{1/2}$, where $B_2 = \max_{v \in \mathcal{N}(\mathcal{N}_2, \delta_0)} \|f(v)\|_H$.

We remark that the condition (iii) is an immediate consequence of the following property:

$$(A) \quad \begin{cases} \text{There exists a constant } \alpha_0 > 0 \text{ independent of } k \text{ such that, for} \\ \text{any } u^0 \in \mathcal{N}_1, \text{ if } T_k^n u^0 \text{ belongs to } \mathcal{N}(u^0, \varepsilon_0), \text{ for } 0 \leq n \leq \beta(k, u^0)/k, \\ \text{with } 0 \leq \beta(k, u^0) \leq \alpha_0 - k, \text{ then } T_k^n u^0 \text{ belongs to } \mathcal{N}(u^0, \varepsilon_0) \text{ for} \\ 0 \leq n \leq (\beta(k, u^0)/k) + 1. \end{cases}$$

Let $u^0 \in \mathcal{N}_1$. We set $u_n = T_k^n u^0$, $\tilde{u}_n = u_n - u^0$ and we assume that, for $0 \leq n \leq \beta(k, u^0)/k$, $T_k^n u^0 \in \mathcal{N}(u^0, \varepsilon_0)$. By (4.1)_k, we have

$$(4.8) \quad \tilde{u}_n - \tilde{u}_{n-1} + kA(\theta \tilde{u}_n + (1 - \theta)\tilde{u}_{n-1}) = kf(u_{n-1}) - kAu^0.$$

Taking the inner product in H of (4.8) by $\tilde{u}_n - \tilde{u}_{n-1}$, we obtain

$$\begin{aligned} \|\tilde{u}_n - \tilde{u}_{n-1}\|_H^2 &+ \frac{k}{2} \|\tilde{u}_n\|_V^2 - \frac{k}{2} \|\tilde{u}_{n-1}\|_V^2 + \frac{k}{2} (2\theta - 1) \|\tilde{u}_n - \tilde{u}_{n-1}\|_V^2 \\ &\leq k(f(u_{n-1}) - f(u^0), \tilde{u}_n - \tilde{u}_{n-1}) \\ &\quad + k(f(u^0), \tilde{u}_n - \tilde{u}_{n-1}) + k(Au^0, \tilde{u}_n - \tilde{u}_{n-1}), \end{aligned}$$

or also,

$$\|\tilde{u}_n\|_V^2 - \|\tilde{u}_{n-1}\|_V^2 \leq kL^2 \|\tilde{u}_{n-1}\|_V^2 + kB_1^2 + (Au^0, \tilde{u}_n - \tilde{u}_{n-1}),$$

where $L > 0$ is the Lipschitz constant of f on \mathcal{N}_3 .

Summation over n yields

$$(4.9) \quad \|\tilde{u}_{m+1}\|_V^2 \leq kL^2 \sum_{n=0}^m \|\tilde{u}_n\|_V^2 + k(m+1)B_1^2 + \|u^0\|_V \|\tilde{u}_{m+1}\|_V,$$

where m is the integral part of $\beta(k, u^0)/k$. Using Lemma 4.2, we infer from (4.9),

$$(4.10) \quad \|\tilde{u}_{m+1}\|_V^2 \leq [B_0^2 + 2k(m+1)B_1^2] \exp(2kL^2(m+1)).$$

Let now α_0 be a positive constant such that

$$(4.11) \quad [B_0^2 + 2\alpha_0 B_1^2] \exp(2L^2 \alpha_0) < \varepsilon_0$$

and choose k_0 such that $0 < k_0 < \alpha_0$. Then one deduces from (4.10) that $\tilde{u}_{m+1} \in \mathcal{N}(u^0, \varepsilon_0)$ if $m+1 \leq \alpha_0/k$, for $0 < k \leq k_0$. Thus, property (A) is shown. As the proof of the condition (iv) uses similar estimates, it is left to the reader.

(3) *Some auxiliary estimates.* We shall estimate

$$k \sum_{n=0}^m \|T(nk)u^0 - T_k^n u^0\|_V^2$$

and

$$\sum_{n=0}^m \|(T((n+1)k)u^0 - T_k^{n+1}u^0) - (T(nk)u^0 - T_k^n u^0)\|_H^2$$

for $0 \leq m \leq \alpha_1/k$, when $T_k^n u^0$ and $T(nk)u^0$ belong to \mathcal{N}_3 for $0 \leq n \leq m$ and $0 \leq nk \leq mk + k_0$, respectively.

We set $t_n = nk$ and $e_n \equiv T_k^n u^0 - T(nk)u^0 \equiv u_n - u(t_n)$. As it was pointed out in Raugel [29, proof of Theorem 2.2], one easily shows that

$$(4.12) \quad \begin{aligned} & k \sum_{n=0}^m \|e_n\|_V^2 + \theta k \|e_{m+1}\|_V^2 - k \sum_{n=0}^m \theta(1-\theta) \|e_{n+1} - e_n\|_V^2 \\ & \leq k \sum_{n=0}^m \|\theta e_{n+1} + (1-\theta)e_n\|_V^2. \end{aligned}$$

From the equations (4.1) and (4.1)_k we infer

$$(4.13) \quad \begin{aligned} & e_{n+1} - e_n + kA(\theta e_{n+1} + (1-\theta)e_n) \\ & = k(f(u_n) - f(u(t_n))) - \int_{t_n}^{t_{n+1}} \left(\frac{du}{dt}(s) - \frac{du}{dt}(t_n) \right) ds \\ & \quad + \theta kA(u(t_n) - u(t_{n+1})). \end{aligned}$$

Taking the inner product in H of (4.13) by $\theta e_{n+1} + (1-\theta)e_n + \gamma_1(e_{n+1} - e_n)$, where $\gamma_1 > 0$, we obtain the following inequality:

$$\begin{aligned} & \frac{1}{2} \|e_{n+1}\|_H^2 - \frac{1}{2} \|e_n\|_H^2 + \frac{1}{2} (2\theta - 1) \|e_{n+1} - e_n\|_H^2 + k \|\theta e_{n+1} + (1-\theta)e_n\|_V^2 \\ & \quad + \gamma_1 \|e_{n+1} - e_n\|_V^2 + \frac{k\gamma_1}{2} \|e_{n+1}\|_V^2 - \frac{k\gamma_1}{2} \|e_n\|_V^2 + \frac{k\gamma_1}{2} (2\theta - 1) \|e_{n+1} - e_n\|_V^2 \\ & \leq kL \|e_n\|_V [(\theta + \gamma_1) \|e_{n+1} - e_n\|_H + \|e_n\|_H] \\ & \quad + \theta k \|u(t_n) - u(t_{n+1})\|_V [\|\theta e_{n+1} + (1-\theta)e_n\|_V + \gamma_1 \|e_{n+1} - e_n\|_V] \\ & \quad + \left\| \int_{t_n}^{t_{n+1}} \left(\frac{du}{dt}(s) - \frac{du}{dt}(t_n) \right) ds \right\|_{V'} [\|\theta e_{n+1} + (1-\theta)e_n\|_V + \gamma_1 \|e_{n+1} - e_n\|_V]. \end{aligned}$$

Using the inequality $ab \leq a^2/2\varepsilon + \varepsilon b^2/2$ several times, we derive from the above estimate:

$$\begin{aligned}
 & \|e_{n+1}\|_H^2 - \|e_n\|_H^2 + k\|\theta e_{n+1} + (1-\theta)e_n\|_V^2 + \gamma_1\|e_{n+1} - e_n\|_H^2 \\
 & \quad + k\gamma_1\|e_{n+1}\|_V^2 - k\gamma_1\|e_n\|_V^2 + \frac{k\gamma_1}{2}(2\theta-1)\|e_{n+1} - e_n\|_V^2 \\
 (4.14) \quad & \leq \left(\frac{k^2\theta^2 L^2}{2\theta-1} + \frac{\gamma_1 k^2 L^2}{2} + k\varepsilon_0 \right) \|e_n\|_V^2 + \frac{kL^2}{\varepsilon_0} \|e_n\|_H^2 \\
 & \quad + k \left(2\theta^2 + \frac{4\gamma_1\theta^2}{2\theta-1} \right) \|u(t_n) - u(t_{n+1})\|_V^2 \\
 & \quad + \left(2 + \frac{4\gamma_1}{2\theta-1} \right) \int_{t_n}^{t_{n+1}} \left\| \frac{du}{dt}(s) - \frac{du}{dt}(t_n) \right\|_{V'}^2 ds,
 \end{aligned}$$

where $\varepsilon_0 > 0$ is a small enough constant.

Summation of (4.14) over n yields

$$\begin{aligned}
 & \|e_{m+1}\|_H^2 + k \sum_{n=0}^m \|\theta e_{n+1} + (1-\theta)e_n\|_V^2 + \gamma_1 \sum_{n=0}^m \|e_{n+1} - e_n\|_H^2 \\
 & \quad + k\gamma_1\|e_{m+1}\|_V^2 + \frac{k\gamma_1}{2}(2\theta-1) \sum_{n=0}^m \|e_{n+1} - e_n\|_V^2 \\
 & \leq k \left(\frac{k\theta^2 L^2}{2\theta-1} + \frac{\gamma_1 k L^2}{2} + \varepsilon_0 \right) \sum_{n=0}^m \|e_n\|_V^2 + \frac{kL^2}{\varepsilon_0} \sum_{n=0}^m \|e_n\|_H^2 \\
 & \quad + k \left(2\theta^2 + \frac{4\gamma_1\theta^2}{2\theta-1} \right) \sum_{n=0}^m \|u(t_n) - u(t_{n+1})\|_V^2 \\
 & \quad + \left(2 + \frac{4\gamma_1}{2\theta-1} \right) \sum_{n=0}^m \int_{t_n}^{t_{n+1}} \left\| \frac{du}{dt}(s) - \frac{du}{dt}(t_n) \right\|_{V'}^2 ds.
 \end{aligned}$$

Now we set $\gamma_1 = \sup(1, 2\theta(1-\theta)/(2\theta-1))$ and we choose $k_0 > 0$ and $\varepsilon_0 > 0$ such that, for $0 < k \leq k_0$, $k\theta^2 L^2/(2\theta-1) + \gamma_1 k L^2/2 + \varepsilon_0 < \frac{1}{2}$. Then, thanks to (4.12), we deduce from the previous inequality that

$$\begin{aligned}
 (4.15) \quad & \|e_{m+1}\|_H^2 + \frac{k}{2} \sum_{n=0}^m \|e_n\|_V^2 + \sum_{n=0}^m \|e_{n+1} - e_n\|_H^2 \\
 & \leq \frac{kL^2}{\varepsilon_0} \sum_{n=0}^m \|e_n\|_H^2 + C(\theta)k \sum_{n=0}^m \|u(t_{n+1}) - u(t_n)\|_V^2 \\
 & \quad + C(\theta) \sum_{n=0}^m \int_{t_n}^{t_{n+1}} \left\| \frac{du}{dt}(s) - \frac{du}{dt}(t_n) \right\|_{V'}^2 ds.
 \end{aligned}$$

Using Lemma 4.2 we infer from (4.15):

$$\begin{aligned}
 (4.16) \quad & \|e_{m+1}\|_H^2 + \frac{k}{2} \sum_{n=0}^m \|e_n\|_V^2 + \sum_{n=0}^m \|e_{n+1} - e_n\|_H^2 \\
 & \leq C(\theta) \exp\left(\frac{kL^2(m+1)}{\varepsilon_0}\right) \\
 & \quad \times \left[k \sum_{n=0}^m \|u(t_{n+1}) - u(t_n)\|_V^2 + \sum_{n=0}^m \int_{t_n}^{t_{n+1}} \left\| \frac{du}{dt}(s) - \frac{du}{dt}(t_n) \right\|_{V'}^2 ds \right].
 \end{aligned}$$

Let us set $B_3 = \max_{v \in \mathcal{N}_3} \|v\|_V$. Then we can write

$$(4.17) \quad \sum_{n=0}^m \|u(t_{n+1}) - u(t_n)\|_V^2 \leq 2B_3^2 + k \int_{t_1}^{t_{m+1}} \left\| \frac{du}{dt} \right\|_V^2 ds.$$

On the other hand, we have

$$(4.18) \quad \begin{aligned} & \sum_{n=0}^m \int_{t_n}^{t_{n+1}} \left\| \frac{du}{dt}(s) - \frac{du}{dt}(t_n) \right\|_{V'}^2 ds \\ & \leq 2k \sup_{t \in [0, t_1]} \left\| \frac{du}{dt}(t) \right\|_{V'}^2 + k^2 \int_{t_1}^{t_{m+1}} \left\| \frac{d^2u}{dt^2} \right\|_{V'}^2 ds, \end{aligned}$$

and, as $A^{-1/2} du/dt = A^{-1/2} f(u) - A^{1/2} u$,

$$(4.19) \quad \sup_{0 \leq t \leq t_{m+1}} \left\| \frac{du}{dt}(t) \right\|_{V'}^2 \leq B_3^2 + \max_{v \in \mathcal{N}_3} \|f(v)\|_H^2.$$

It now remains to bound the quantities

$$\int_{t_1}^{t_{m+1}} k \left\| \frac{du}{dt} \right\|_V^2 ds \quad \text{and} \quad \int_{t_1}^{t_{m+1}} k \left\| \frac{d^2u}{dt^2} \right\|_{V'}^2 ds.$$

By Henry [17, p. 71], there exist two constants $K_0 > 0$ and $K_1(\mathcal{N}_3) > 0$ such that, for $0 \leq t \leq mk + k_0$,

$$(4.20) \quad t \left\| \frac{du}{dt}(t) \right\|_V + t^{1/2} \left\| \frac{du}{dt}(t) \right\|_H \leq K_1(\mathcal{N}_3) e^{K_0(\alpha_1 + k_0)}.$$

Since the inequalities (3.25) and (4.20) hold, $f'(u)t du/dt + du/dt$ belongs to $L^2([0, t_{m+1}]; H)$ and one easily proves that the function $t du/dt$ satisfies the equation

$$(4.21) \quad \begin{cases} \text{(i)} \quad \left(\frac{d}{dt} \left(t \frac{du}{dt} \right), \phi \right) + a \left(t \frac{du}{dt}, \phi \right) = \left(f'(u) t \frac{du}{dt}, \phi \right) + \left(\frac{du}{dt}, \phi \right) \\ \text{for } \phi \in V; \\ \text{(ii)} \quad \left(t \frac{du}{dt} \right)_{/t=0} = 0. \end{cases}$$

Hence, $t du/dt$ belongs to the space $H^1([0, t_{m+1}]; H)$. For $t > 0$, Eq. (4.21)(i) can also be written as

$$(4.22) \quad \left(t \frac{d^2u}{dt^2}, \phi \right) + a \left(t \frac{du}{dt}, \phi \right) = \left(f'(u) t \frac{du}{dt}, \phi \right) \quad \text{for any } \phi \in V.$$

We set $\phi = du/dt$ in (4.22); then, after an integration from 0 to t_{m+1} , we obtain

$$(4.23) \quad \begin{aligned} & \frac{1}{2} \left(t \left\| \frac{du}{dt} \right\|_H^2 \right) (t_{m+1}) + \gamma \int_0^{t_{m+1}} t \left\| \frac{du}{dt} \right\|_V^2 dt \\ & \leq \frac{1}{2} \int_0^{t_{m+1}} \left\| f'(u) t \frac{du}{dt} \right\|_H^2 dt + \frac{3}{2} \int_0^{t_{m+1}} \left\| \frac{du}{dt} \right\|_H^2 dt \\ & \quad + \left(\frac{1}{2} t \left\| \frac{du}{dt} \right\|_H^2 \right) (0). \end{aligned}$$

Since

$$\int_0^{t_{m+1}} \left\| f'(u) t \frac{du}{dt} \right\|_H^2 dt \leq \sup_{u \in \mathcal{N}_3} \|f'(u)\|_{\mathcal{L}(V;H)}^2 \int_0^{t_{m+1}} t^2 \left\| \frac{du}{dt} \right\|_V^2 dt,$$

we deduce from (4.23), by using (3.25) and (4.20), that

$$(4.24) \quad \int_0^{t_{m+1}} t \left\| \frac{du}{dt} \right\|_V^2 dt \leq K_2(\mathcal{N}_3) e^{K_3(\alpha_1 + k_0)},$$

where $K_2(\mathcal{N}_3)$ and K_3 are two positive constants.

If we set $\phi = A^{-1} d^2u/dt^2$ in (4.22), we obtain

$$\begin{aligned} & \frac{1}{2} \int_0^{t_{m+1}} t \left\| A^{-1/2} \frac{d^2u}{dt^2} \right\|_H^2 dt + \frac{1}{2} \left(t \left\| \frac{du}{dt} \right\|_H^2 \right) (t_{m+1}) \\ & \leq \frac{1}{2} \int_0^{t_{m+1}} \left\| A^{-1/2} f'(u) t^{1/2} \frac{du}{dt} \right\|_H^2 dt + \int_0^{t_{m+1}} \left\| \frac{du}{dt} \right\|_H^2 dt \\ & \quad + \frac{1}{2} \left(t \left\| \frac{du}{dt} \right\|_H^2 \right) (0), \end{aligned}$$

which implies, thanks to (3.25), (4.20) and (4.24), that

$$(4.25) \quad \int_0^{t_{m+1}} t \left\| \frac{d^2u}{dt^2}(t) \right\|_V^2 dt \leq K_4(\mathcal{N}_3) e^{K_5(\alpha_1 + k_0)},$$

where $K_4(\mathcal{N}_3)$ and K_5 are positive constants.

Finally, from (4.16), (4.17), (4.18), (4.19), (4.24) and (4.25), we infer

$$(4.26) \quad \|e_{m+1}\|_H^2 + \frac{k}{2} \sum_{n=0}^m \|e_n\|_V^2 + \sum_{n=0}^m \|e_{n+1} - e_n\|_H^2 \leq kK_6(\mathcal{N}_3) e^{K_7(\alpha_1 + k_0)},$$

where $K_6(\mathcal{N}_3)$ and K_7 are two positive constants.

(4) *Estimate of $\|T(nk)u^0 - T_k^n u^0\|_V$ for $\alpha_0/k \leq n \leq m+1$, when $T_k^n u^0$ and $T(nk)u^0$ belong to \mathcal{N}_3 for $0 \leq n \leq m$ and $0 \leq nk \leq mk + k_0$, respectively, where $\alpha_0/k < m \leq \alpha_1/k$. To this end, we at first estimate the term $\|t_n(T(t_n)u^0 - T_k^n u^0)\|_V$ for $0 \leq n \leq m$. Formula (4.1)_k gives*

$$(4.27) \quad \begin{aligned} & t_{n+1}u_{n+1} - t_n u_n + kA(\theta t_{n+1}u_{n+1} + (1-\theta)t_n u_n) \\ & = kt_n f(u_n) + ku_{n+1} + \theta k^2 Au_{n+1}. \end{aligned}$$

We set $\bar{e}_n = t_n(u_n - u(t_n))$. From (4.27) and from the equation (4.1) we deduce

$$(4.28) \quad \begin{aligned} & \bar{e}_{n+1} - \bar{e}_n + kA(\theta \bar{e}_{n+1} + (1-\theta)\bar{e}_n) \\ & = kt_n(f(u_n) - f(u(t_n))) - \int_{t_n}^{t_{n+1}} \left[\frac{d}{ds}(su(s)) - \left(\frac{d}{ds}(su(s)) \right)_{/s=t_n} \right] ds \\ & \quad + \theta kA(t_n u(t_n) - t_{n+1}u(t_{n+1})) + ke_n + k(u_{n+1} - u_n) + \theta k^2 Au_{n+1}. \end{aligned}$$

Taking the inner product in H of (4.28) by $\bar{e}_{n+1} - \bar{e}_n$, we obtain

$$\begin{aligned} & \|\bar{e}_{n+1} - \bar{e}_n\|_H^2 + \frac{k}{2} \|\bar{e}_{n+1}\|_V^2 - \frac{k}{2} \|\bar{e}_n\|_V^2 + \frac{k}{2} (2\theta - 1) \|\bar{e}_{n+1} - \bar{e}_n\|_V^2 \\ & \leq k \|\bar{e}_{n+1} - \bar{e}_n\|_H [L \|\bar{e}_n\|_V + \|e_n\|_H + \|u_{n+1} - u_n\|_H] \\ & \quad + k \|\bar{e}_{n+1} - \bar{e}_n\|_V [\theta k \|u_{n+1}\|_V + \theta \|t_n u(t_n) - t_{n+1}u(t_{n+1})\|_V] \\ & \quad + \frac{1}{k} \left\| \int_{t_n}^{t_{n+1}} \left[\frac{d}{ds}(su(s)) - \left(\frac{d}{ds}(su(s)) \right)_{/s=t_n} \right] ds \right\|_V, \end{aligned}$$

or also,

$$\begin{aligned} \|\bar{e}_{n+1}\|_V^2 - \|\bar{e}_n\|_V^2 &\leq 2kL^2\|\bar{e}_n\|_V^2 + 2k\|e_n\|_H^2 + 2k\|u_{n+1} - u_n\|_H^2 \\ &\quad + \frac{3}{2\theta - 1} \left[k^2\|u_{n+1}\|_V^2 + k \int_{t_n}^{t_{n+1}} \left\| \frac{d}{ds}(su(s)) \right\|_V^2 ds \right. \\ &\quad \left. + k \int_{t_n}^{t_{n+1}} \left\| \frac{d^2}{ds^2}(su(s)) \right\|_{V'}^2 ds \right]. \end{aligned}$$

Summing the previous inequality over n and applying Lemma 4.2, we have

$$\begin{aligned} &\max_{0 \leq n \leq m+1} \|\bar{e}_{n+1}\|_V^2 \\ &\leq C \exp(2kL^2(m+1)) \left[k \sum_{n=0}^m \|e_n\|_H^2 + k \sum_{n=0}^m \|u_{n+1} - u_n\|_H^2 \right. \\ (4.29) \quad &\quad \left. + k^2 \sum_{n=0}^m \|u_{n+1}\|_V^2 + k \int_0^{t_{m+1}} \left\| \frac{d}{ds}(su(s)) \right\|_V^2 ds \right. \\ &\quad \left. + k \int_0^{t_{m+1}} \left\| \frac{d^2}{ds^2}(su(s)) \right\|_{V'}^2 ds \right]. \end{aligned}$$

But

$$(4.30) \quad k \sum_{n=0}^m \|u_{n+1} - u_n\|_H^2 \leq 2k \sum_{n=0}^m \|e_{n+1} - e_n\|_H^2 + k^2 \int_0^{t_{m+1}} \left\| \frac{du}{dt} \right\|_H^2 ds$$

and

$$(4.31) \quad k^2 \sum_{n=0}^m \|u_{n+1}\|_V^2 \leq k^2(m+1)B_3^2.$$

Finally, we derive from (4.29), (4.30), (4.31), (4.26), as well as from (3.25), (4.20) and (4.25), that

$$\max_{0 \leq n \leq m+1} \|\bar{e}_{n+1}\|_V \leq k^{1/2} K_8(\mathcal{N}_3) e^{K_9(\alpha_1 + k_0)},$$

where $K_8(\mathcal{N}_3)$ and K_9 are positive constants.

Hence, we have

$$(4.32) \quad \max_{\alpha_0/k \leq n \leq m+1} \|T(nk)u^0 - T_k^n u^0\|_V \leq \frac{k^{1/2}}{\alpha_0} K_8(\mathcal{N}_3) e^{K_9(\alpha_1 + k_0)}.$$

This completes the proof of Theorem 4.1. \square

Remark 4.2. If f is locally Lipschitz continuous from V into H and from H into V' , we can improve the estimate (4.32) (and the proof is shorter). For estimates in the case where f is globally Lipschitz continuous from H into H , we refer the reader to Crouzeix and Thomée [8].

Remark 4.3. Now we consider a discretization in space and time of the equation (4.1). More precisely, if $(V_h)_h$ are the spaces given in Subsection 3.1, we define an approximation $u_n^h \in V_h$ of the solution u of (4.1) at time t_n by the recursion formula

$$(4.1)_k^h \quad \begin{cases} u_{n+1}^h = (1 - (1 - \theta)kA_h)(1 + \theta kA_h)^{-1}u_n^h + k(1 + \theta kA_h)^{-1}Q_h f(u_n^h), \\ u_0^h = u_h^0 \in V_h \end{cases}$$

(where A_h and Q_h are given in Subsection 3.1).

Then in the same way as above, one proves that $(4.1)_k^h$ gives rise to a dynamical system T_k^h which admits an attractor \mathcal{A}_k^h . And $\delta_V(\mathcal{A}_k^h, \mathcal{A}) \rightarrow 0$ as h and k tend to 0.

Furthermore, if we are in the situation described in Subsection 3.2 and if $kh^{-2m} \leq C$, where C is a positive constant, one can define a dynamical system \tilde{T}_k^h which admits an attractor $\tilde{\mathcal{A}}_k^h$ in Y^α and $\delta_{Y^\alpha}(\tilde{\mathcal{A}}_k^h, \mathcal{A}) \rightarrow 0$ as h and k tend to 0.

5. A Remark on the Two-Dimensional Navier-Stokes Equations. Let Ω be a regular, bounded domain in \mathbf{R}^2 . The Navier-Stokes equations for the velocity $u(x, t) = (u_1(x, t), u_2(x, t))$ and the pressure $p(x, t)$ are

$$(5.1) \quad \begin{cases} \frac{\partial u}{\partial t} - \nu \Delta u + \sum_{i=1}^2 u_i \frac{\partial u}{\partial x_i} + \text{grad } p = F & \text{in } \Omega \times \mathbf{R}_+, \\ \text{div } u = 0 & \text{in } \Omega \times \mathbf{R}_+, \\ u = 0 & \text{on } \partial\Omega \times \mathbf{R}_+, \\ u(x, 0) = u_0(x) & \text{in } \Omega, \end{cases}$$

where F and u_0 are given and $\nu > 0$ is the kinematic viscosity. Let us denote by $\mathbf{H}^j(\Omega)$ the space $(H^j(\Omega))^2$ for $j = 1$ or 2 and by $\mathbf{L}^2(\Omega)$ the space $(L^2(\Omega))^2$. We consider the space

$$\mathcal{V} = \{ \phi \in (C_0^\infty(\Omega))^2; \text{div } \phi = 0 \}$$

and denote by H and V the closures of \mathcal{V} in $\mathbf{L}^2(\Omega)$ and $\mathbf{H}^1(\Omega)$, respectively. The spaces H and V are provided with the inner products

$$(u, v) = \sum_{j=1}^2 \int_{\Omega} u_j v_j \, dx, \quad \text{and} \quad ((u, v)) = \sum_{j,k=1}^2 \int_{\Omega} \frac{\partial u_k}{\partial x_j} \frac{\partial v_k}{\partial x_j} \, dx,$$

respectively, where $x = (x_1, x_2)$.

We also set $|u| = (u, u)^{1/2}$ and $\|u\| = ((u, u))^{1/2}$ for u in H and V , respectively.

Let us denote by P the orthogonal projection of $\mathbf{L}^2(\Omega)$ onto H . We define $A = -P\Delta$ to be the operator with domain $D(A) = \mathbf{H}^2(\Omega) \cap V$ acting in H and use the same notation for its extension to an operator from V into V' . Since A^{-1} is a compact selfadjoint linear operator in H , the spectrum of A consists of an infinite sequence

$$0 < \lambda_1 \leq \lambda_2 \leq \dots$$

of eigenvalues (counted according to their multiplicities), $\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$, and there exists an orthonormal basis $\{\phi_n\}_{n \geq 1}$ of H such that

$$A\phi_n = \lambda_n \phi_n, \quad n = 1, 2, \dots$$

For any $N \geq 1$ we denote by P_N the orthogonal projection in H (and in $V, V', D(A)$) onto the space V_N spanned by $\phi_1, \phi_2, \dots, \phi_N$. For $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $\mathbf{H}^1(\Omega)$ we define $B(u, v) \in V'$ by

$$(5.2) \quad (B(u, v), w) = \sum_{j,k=1}^2 \int_{\Omega} u_j \frac{\partial v_k}{\partial x_j} w_k \, dx \quad \forall w \in V.$$

Then B is a bilinear continuous operator from $\mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega)$ into V' and this operator can be extended as an operator from $\mathbf{H}^{m_1}(\Omega) \times \mathbf{H}^{m_2}(\Omega)$ into V' or H , for appropriate values of m_1 and m_2 (see Témam [32] for instance).

Using the above notations, it can be shown that (5.1) is equivalent to the following initial value problem

$$(5.3) \quad \begin{cases} du/dt + \nu Au + B(u, u) = f & \text{in } H, \\ u(0) = u_0, \end{cases}$$

where we assume that $f(x) = PF(x)$ and u_0 belong to H and V , respectively (see [32] for further details). Let us point out that f does not depend on t .

Now we introduce the map $T(t): V \rightarrow V$, $t \geq 0$, defined by $T(t)u_0 = u(t)$, where $u(t)$ is the solution of (5.3). It is well known that $T(t)u_0$ exists for any $t \geq 0$ and any $u_0 \in V$ and that $T(t)$ is a C^0 -semigroup on V (see Ladyzhenskaya [23], [24], for instance). In the same papers she also showed that $T(t)u_0$ has its $\overline{\lim}$ as $t \rightarrow +\infty$ bounded by a constant independent of the initial data, i.e., $T(t)$ is point dissipative. Since $T(t)$ is compact for $t > 0$, we deduce from a result of Billotti and LaSalle [2] that $T(t)$ admits a compact attractor \mathcal{A} which attracts bounded sets of V (see also [12]).

Now let us consider the following differential system on the space V_N spanned by $\phi_1, \phi_2, \dots, \phi_N$:

$$(5.3)_N \quad \begin{cases} du_N/dt + \nu Au_N + P_N B(u_N, u_N) = P_N f(x), \\ u_N(0) = u_{0N}, \end{cases}$$

where $u_{0N} \in V_N$. We introduce the map $T_N(t): V_N \rightarrow V_N$, $t \geq 0$, defined by $T_N(t)u_{0N} = u_N(t)$, where $u_N(t)$ is the solution of (5.3)_N. As above, $T_N(t)$ is a C^0 -semigroup on V_N (see Témam [32] for instance). In [32, Section 14.2], it is also shown that $T_N(t)u_{0N}$ has its $\overline{\lim}$ as $t \rightarrow +\infty$ bounded by a constant independent of the initial data and of N . Thus, by [2], $T_N(t)$ admits a compact attractor \mathcal{A}_N which attracts bounded sets of V_N . But thanks to Theorem 2.4 we obtain the following more precise result (for related results, see Constantin, Foias and Témam [6]).

THEOREM 5.1. *For any $N \geq 1$, T_N admits a compact attractor \mathcal{A}_N which attracts bounded sets of V_N . Moreover, $\delta_V(\mathcal{A}_N, \mathcal{A}) \rightarrow 0$ as $N \rightarrow +\infty$.*

Theorem 5.1 is a straightforward consequence of Lemma 2.1. In order to prove that $\|T(t)u_N - T_N(t)u_N\|_V$ satisfies the conditions (2.2) and (2.3), we use relations and inequalities which are similar to those contained in [32, Section 3.1]. In particular, we use the Young inequality, Gronwall's lemma technique (see [32, Formulas (3.10), (3.12)]), Lemmas 11.1 and 14.3 of [32] and the following inequality:

$$(5.4) \quad \begin{aligned} |(B(u, v), w)| &\leq C_1 |u|^{1/2} \|u\|^{1/2} \|v\|^{1/2} |Av|^{1/2} |w| \\ &\forall u \in V, v \in D(A), w \in H. \end{aligned}$$

The proof of Theorem 5.1 is given in Hale, Lin and Raugel [15].

Remark 5.1. The same kind of proof shows that if \mathcal{A}_{ν_0} is the attractor associated with (5.3) for $\nu = \nu_0$, then $\delta_V(\mathcal{A}_\nu, \mathcal{A}_{\nu_0}) \rightarrow 0$ as $\nu \rightarrow \nu_0$, where $\nu_0 \neq 0$.

Remark 5.2. The generalization of Theorem 5.1 to an approximation of the Navier-Stokes equations by a finite element method seems technically involved. In [19], Heywood and Rannacher have given uniform in time error estimates for the approximation of the Navier-Stokes equations by a finite element method when the solution is exponentially stable (see [18], [19] for the definition). The referee has called our attention to this recent paper ([19]) of Heywood and Rannacher and has

pointed out that our method of proof of Lemma 2.1 has some analogy with their proof of Theorem 3.2 in [19].

6. Approximation of the Damped Wave Equation. Let Ω be a bounded domain in \mathbf{R}^3 , α be a positive constant and consider the equation

$$(6.1) \quad \begin{cases} \frac{\partial^2 u}{\partial t^2} + 2\alpha \frac{\partial u}{\partial t} - \Delta u = -f(u) - g(x) & \text{in } \Omega \times (0, \infty), \\ u = 0 & \text{on } \partial\Omega, \\ \left(u, \frac{\partial u}{\partial t} \right) \Big|_{t=0} = (\phi, \psi), \end{cases}$$

where g belongs to $L^2(\Omega)$ and (ϕ, ψ) belongs to the space $X \equiv H_0^1(\Omega) \times L^2(\Omega)$. We assume that the boundary $\partial\Omega$ of Ω is smooth enough or that Ω is a convex domain. Furthermore, we suppose that $f \in C^3(\mathbf{R})$ and that there are constants $\gamma \geq 0$, $C_1 > 0$, with $0 \leq \gamma < 2$, such that

$$(6.2) \quad \begin{cases} |f(v)| \leq C_1(|v|^{1+\gamma} + 1), & |f'(v)| \leq C_1(|v|^\gamma + 1), \\ |f''(v)| \leq C_1(|v| + 1), & |f'''(v)| \leq C_1 \text{ for all } v \in \mathbf{R}. \end{cases}$$

Inequalities (6.2) imply that the map $f: \phi \in H_0^1(\Omega) \rightarrow f(\phi(x)) \in L^2(\Omega)$ is a compact C^2 -mapping from $H_0^1(\Omega)$ into $L^2(\Omega)$. Henceforth, we equip the space X with the norm

$$(6.3) \quad \|(\phi, \psi)\|_X = (\|\phi\|_{H_0^1(\Omega)}^2 + \|\psi\|_{L^2(\Omega)}^2)^{1/2} \quad \forall (\phi, \psi) \in X.$$

As it was proved in Babin and Vishik [1], for any $(\phi, \psi) \in X$, Problem (6.1) has a unique solution $u(t)$, for $t \geq 0$, and the pair $(u, \partial u/\partial t)$ belongs to $C^0([0, +\infty); X)$. Furthermore, if we set $T(t)(\phi, \psi) = (u(t), \partial u(t)/\partial t)$, for $t \geq 0$, then $T(t): X \rightarrow X$, $t \geq 0$, is a C^0 -semigroup on X .

Now suppose there is a constant $C > 0$ so that f satisfies

$$(6.4) \quad f(v)v > -C, \quad f'(v) > -C \quad \text{for all } v \in \mathbf{R}.$$

Let us introduce the Liapunov functional V given by

$$V(\phi, \psi) = \int_{\Omega} \left[\frac{1}{2} |\nabla \phi(x)|^2 + \frac{1}{2} \psi^2(x) + F(\phi(x)) + g(x)\phi(x) \right] dx,$$

for all $(\phi, \psi) \in X$, where $F(v) = \int_0^v f(s) ds$. It was proved in Babin and Vishik [1] that

$$(6.5) \quad \begin{cases} V(\phi, \psi) \geq \frac{1}{2} \|\psi\|_{L^2(\Omega)}^2 + \|\phi\|_{H_0^1(\Omega)}^2 - C_2, \\ V(\phi, \psi) \leq \frac{1}{2} \|\psi\|_{L^2(\Omega)}^2 + C_3 \|\phi\|_{H_0^1(\Omega)}^4 + C_4, \end{cases}$$

where C_2, C_3, C_4 are some fixed positive constants, and that, for $t \geq \tau$ and for any solution u of (6.1),

$$(6.6) \quad V \left(u(t, \cdot), \frac{\partial u}{\partial t}(t, \cdot) \right) - V \left(u(\tau, \cdot), \frac{\partial u}{\partial t}(\tau, \cdot) \right) = -2\alpha \int_{\tau}^t \int_{\Omega} \left[\frac{\partial u}{\partial t}(s, x) \right]^2 dx ds.$$

The properties (6.5), (6.6) imply that the orbits of bounded sets are bounded. In particular, there exist two functions $C_0(R)$ and $C_1(R)$ of R such that, if

$$(6.7) \quad \|(\phi, \psi)\|_X \leq R,$$

then

$$(6.8) \quad V(T(t)(\phi, \psi)) \leq V(\phi, \psi) \leq C_0(R) \quad \forall t \in R,$$

and

$$(6.9) \quad \|T(t)(\phi, \psi)\|_X \leq C_1(R) \quad \forall t \in R.$$

Moreover, it was shown in Hale [12] that $T(t)$ is point dissipative and is an α -contraction. Therefore, in view of a result of Massatt [27], $T(t)$ admits a compact attractor \mathcal{A} in X which attracts bounded sets of X (see Hale [12, Theorem 6.1]).

6.1. Approximation By a Special Projection Method. Let us recall that the spectrum of the operator $-\Delta$ with domain $D(-\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$ consists of an infinite sequence

$$0 < \lambda_1 \leq \lambda_2 \leq \dots$$

of eigenvalues (counted according to their multiplicities), $\lambda_n \rightarrow +\infty$ as $n \rightarrow +\infty$, and that there exists an orthonormal basis $\{w_n\}_{n \geq 1}$ of $L^2(\Omega)$ such that

$$(6.10) \quad -\Delta w_n = \lambda_n w_n.$$

Note that $\{\lambda_n^{-1/2} w_n\}_{n \geq 1}$ is an orthonormal basis of $H_0^1(\Omega)$. For any $N \geq 1$ we denote by P_N the orthogonal projection in $L^2(\Omega)$ (and in $H_0^1(\Omega)$) onto the space V_N spanned by w_1, w_2, \dots, w_N , and we consider the following equation in V_N :

$$(6.1)_N \quad \begin{cases} \frac{\partial^2 u_N}{\partial t^2} + 2\alpha \frac{\partial u_N}{\partial t} - \Delta u_N = -P_N f(u_N) - P_N g(x), \\ \left(u_N, \frac{\partial u_N}{\partial t} \right) \Big|_{t=0} = (\phi_N, \psi_N), \end{cases}$$

where (ϕ_N, ψ_N) belongs to the space $X_N = V_N \times V_N$. We can prove, as for Problem (6.1), that, for any (ϕ_N, ψ_N) in X_N , the equation (6.1)_N has a unique solution $u_N(t)$ for $t \geq 0$. Moreover, if we set $T_N(t)(\phi_N, \psi_N) = (u_N(t), \partial u_N(t)/\partial t)$ for $t \geq 0$, then $T_N(t): X_N \rightarrow X_N$, $t \geq 0$, is a C^0 -semigroup on X_N .

THEOREM 6.1. *For any $N \geq 1$, T_N admits a compact attractor \mathcal{A}_N which attracts bounded sets of X_N . Moreover, $\delta_X(\mathcal{A}_N, \mathcal{A}) \rightarrow 0$ as $N \rightarrow +\infty$.*

Proof. (1) We at once verify that, for $t \geq \tau$, for any solution u_N of Eq. (6.1)_N,

$$(6.11) \quad \begin{aligned} & V \left(u_N(t, \cdot), \frac{\partial u_N}{\partial t}(t, \cdot) \right) - V \left(u_N(\tau, \cdot), \frac{\partial u_N}{\partial t}(\tau, \cdot) \right) \\ &= -2\alpha \int_{\tau}^t \int_{\Omega} \left(\frac{\partial u_N}{\partial t}(s, x) \right)^2 dx ds. \end{aligned}$$

The estimates (6.5) imply that the orbits of bounded sets are bounded independently of N . In particular, $T_N(t)(\phi_N, \psi_N)$ satisfies the estimates (6.8) and (6.9), for any (ϕ_N, ψ_N) satisfying (6.7). As $T_N(t)$ is compact, the orbit through (ϕ_N, ψ_N) is precompact and its ω -limit set must be an invariant set. Relation (6.11) implies that its ω -limit set belongs to the set E_N of the equilibrium points. Using the condition (6.4), one easily proves that there exists a constant $r_0 > 0$ such that

$$(6.12) \quad \forall N \geq 1, \quad E_N \subset B_X(r_0),$$

where, for any $r > 0$, $B_X(r) = \{(\phi, \psi) \in X: \|(\phi, \psi)\|_X < r\}$. Let us also set $B_{X_N}(r) = B_X(r) \cap X_N$. Then, for $r_1 = 2r_0$, the ball $B_X(r_1)$ attracts all points

of X_N (i.e., for any $(\phi_N, \psi_N) \in X_N$, there exists $t_N \geq 0$ such that, for $t \geq t_N$, $T_N(t)(\phi_N, \psi_N) \in B_{X_N}(r_1)$). We remark that the orbit of $B_{X_N}(r_1)$ is included in $B_{X_N}(C_1(r_1))$, where $C_1(r_1)$ is given by (6.9), and that $B_{X_N}(C_1(r_1))$ attracts a neighborhood of any point and, hence, all compact sets of X_N . We now set $R_0 = C_1(C_1(r_1))$. Arguing as in Hale [11, Theorem 2.1], one finally shows that $T_N(t)$ admits a compact attractor \mathcal{A}_N which attracts bounded sets of X_N and is included in the ball $B_X(R_0) \cap X_N$.

(2) In order to prove that $\delta_X(\mathcal{A}_N, \mathcal{A}) \rightarrow 0$ as $N \rightarrow +\infty$, we show that the hypotheses of Lemma 2.1 hold. Let $\mathcal{N}_1 \equiv B_X(R_1)$ be a neighborhood of \mathcal{A} . We shall prove that $T_N(t)$ approximates $T(t)$ on \mathcal{N}_1 uniformly on compact sets of $[0, +\infty)$. Let t_1 be any positive real number. We first estimate

$$\left\| \left(u(t) - P_N u(t), \frac{\partial u}{\partial t}(t) - \frac{\partial P_N u}{\partial t}(t) \right) \right\|_X$$

for $0 \leq t \leq t_1$, where $(u(t), \partial u(t)/\partial t) = T(t)(\phi_N, \psi_N)$ and $(\phi_N, \psi_N) \in \mathcal{N}_1 \cap X_N$. We have

$$(6.13) \quad \begin{aligned} & \frac{\partial^2}{\partial t^2}(u - P_N u) + 2\alpha \frac{\partial}{\partial t}(u - P_N u) - \Delta(u - P_N u) \\ & = -(I - P_N)f(u) - (I - P_N)g(x). \end{aligned}$$

Taking the inner product in $L^2(\Omega)$ of (6.13) by $\partial(u - P_N u)/\partial t$, we get after an integration from 0 to t ,

$$(6.14) \quad \begin{aligned} & \left\| \frac{\partial}{\partial t}(u - P_N u)(t) \right\|_{L^2(\Omega)}^2 + \|u(t) - P_N u(t)\|_{H_0^1(\Omega)}^2 \\ & \leq \frac{t_1}{\alpha} \left(\sup_{s \in [0, t_1]} \|(I - P_N)f(u(s))\|_{L^2(\Omega)}^2 + \|(I - P_N)g(x)\|_{L^2(\Omega)}^2 \right). \end{aligned}$$

Since f is a compact mapping from $H_0^1(\Omega)$ into $L^2(\Omega)$ and $u(s)$, $0 \leq s \leq t$, belongs to the bounded set $\overline{B}(C_1(R_1)) = \{v \in H_0^1(\Omega) : \|v\|_{H_0^1(\Omega)} \leq C_1(R_1)\}$, we deduce from (6.14) that, for $0 \leq t \leq t_1$,

$$(6.15) \quad \left\| \frac{\partial}{\partial t}(u - P_N u)(t) \right\|_{L^2(\Omega)}^2 + \|u(t) - P_N u(t)\|_{H_0^1(\Omega)}^2 \leq \eta_1(N, t_1, C_1(R_1)),$$

where

$$(6.16) \quad \lim_{N \rightarrow +\infty} \eta_1(N, t_1, C_1(R_1)) = 0.$$

Now we estimate

$$\left\| \left(P_N u(t) - u_N(t), \frac{\partial}{\partial t}(P_N u(t) - u_N(t)) \right) \right\|_X$$

for $0 \leq t \leq t_1$, where $(u_N(t), \partial u_N(t)/\partial t) = T_N(t)(\phi_N, \psi_N)$. The function $u_N - P_N u$ satisfies the equation

$$(6.17) \quad \frac{\partial^2}{\partial t^2}(u_N - P_N u) + 2\alpha \frac{\partial}{\partial t}(u_N - P_N u) - \Delta(u_N - P_N u) = P_N(f(u) - f(u_N)).$$

Taking the inner product in $L^2(\Omega)$ of (6.17) with $\partial(u_N - P_N u)/\partial t$, we obtain

$$(6.18) \quad \begin{aligned} & \frac{1}{2} \frac{\partial}{\partial t} \left(\left\| \frac{\partial}{\partial t} (u_N - P_N u) \right\|_{L^2(\Omega)}^2 \right) + \frac{1}{2} \frac{\partial}{\partial t} (\|u_N - P_N u\|_{H_0^1(\Omega)}^2) \\ & \leq \frac{L^2}{2\alpha} \|u_N - P_N u\|_{H_0^1(\Omega)}^2 + \frac{L^2}{2\alpha} \|u - P_N u\|_{H_0^1(\Omega)}^2, \end{aligned}$$

where $L > 0$ is the Lipschitz constant of f in the ball $\bar{B}(C_1(R_1))$. Now using Gronwall's lemma, we derive from (6.18) as well as from (6.15) that, for $0 \leq s \leq t$,

$$(6.19) \quad \begin{aligned} & \left\| \frac{\partial}{\partial t} (u_N - P_N u)(t) \right\|_{L^2(\Omega)}^2 + \|u_N(t) - P_N u(t)\|_{H_0^1(\Omega)}^2 \\ & \leq e^{t_1 L^2/2\alpha} \frac{L^2}{2\alpha} \eta_1(N, t_1, C_1(R_1)). \end{aligned}$$

The estimates (6.15), (6.16) and (6.19) show that $T_N(t)$ approximates $T(t)$ on \mathcal{N}_1 uniformly on compact sets of $[0, +\infty)$. \square

6.2. A More General Galerkin Method. Let $h > 0$ be a real parameter which will tend to 0 and $(V_h)_h$ be a family of finite-dimensional subspaces of $H_0^1(\Omega)$. We denote by $[\cdot, \cdot]$ the inner product of $L^2(\Omega)$ and by $a(\cdot, \cdot)$ the inner product of $H_0^1(\Omega)$, i.e.,

$$(6.20) \quad \forall v \in H_0^1(\Omega), \forall w \in H_0^1(\Omega), \quad a(v, w) = \int_{\Omega} \nabla v \nabla w \, dx.$$

As in Subsection 3.1, we denote by $Q_h \in \mathcal{L}(L^2(\Omega); V_h)$ and $P_h \in \mathcal{L}(H_0^1(\Omega); V_h)$ the orthogonal projectors on V_h in the spaces $L^2(\Omega)$ and $H_0^1(\Omega)$, respectively. We also introduce the operator $A_h \in \mathcal{L}(V_h; V_h)$ defined by

$$\forall v_h \in V_h, \quad (A_h w_h, v_h) = a(w_h, v_h) \quad \text{for } w_h \in V_h.$$

We consider the following equation in V_h :

$$(6.1)_h \quad \begin{cases} \frac{\partial^2 u_h}{\partial t^2} + 2\alpha \frac{\partial u_h}{\partial t} + A_h u_h = -Q_h f(u_h) - Q_h g(x), \\ \left(u_h, \frac{\partial u_h}{\partial t} \right) \Big|_{t=0} = (\phi_h, \psi_h), \end{cases}$$

where (ϕ_h, ψ_h) belongs to the space $X_h = V_h \times V_h$. As in Subsection 6.1, we introduce the map $T_h(t): X_h \rightarrow X_h$, for $t \geq 0$, defined by

$$T_h(\phi_h, \psi_h) = (u_h(t), \partial u_h(t)/\partial t),$$

where u_h is the solution of (6.1)_h. So we obtain a C^0 -semigroup on X_h . As in Subsection 3.1, we need some additional hypotheses on the spaces $(V_h)_h$:

there exists a constant $K_0 > 0$, independent of h , such that, for any $h > 0$,

$$(6.21)(i) \quad \|Q_h\|_{\mathcal{L}(H_0^1(\Omega); H_0^1(\Omega))} \leq K_0$$

and

there exist two constants $K_1 > 0$ and $\theta > 0$, independent of h , such that, for any w in $H_0^1(\Omega)$,

$$(6.21)(ii) \quad \|w - P_h w\|_{L^2(\Omega)} + \|w - Q_h w\|_{L^2(\Omega)} \leq K_1 h^\theta \|w\|_{H_0^1(\Omega)}.$$

(Usually, θ is taken equal to 1.) Finally, we introduce the Hilbert space $Y \equiv L^2(\Omega) \times H^{-1}(\Omega)$, normed by $\|(\phi, \psi)\|_Y = (\|\phi\|_{L^2}^2 + \|\psi\|_{H^{-1}(\Omega)}^2)^{1/2}$. Below we denote by $[\cdot, \cdot]$ the inner product in $L^2(\Omega)$. Now we are able to prove the following result.

THEOREM 6.2. *For any $h > 0$, T_h admits a compact attractor \mathcal{A}_h which attracts bounded sets of X_h and is contained in the ball $B_X(R_0) \cap X_h$, where R_0 is a constant independent of h . Moreover, $\delta_Y(\mathcal{A}_h, \mathcal{A}) \rightarrow 0$ as $h \rightarrow 0$.*

Remark 6.1. In Section 3 we proved that $\delta_V(\mathcal{A}_h, \mathcal{A}) \rightarrow 0$ as $h \rightarrow 0$. Here, we can no longer prove that $\delta_X(\mathcal{A}_h, \mathcal{A}) \rightarrow 0$ as $h \rightarrow 0$, because $T(t)$ has no longer a smoothing action.

Proof of Theorem 6.2. (1) First we show in the same way as in the proof of Theorem 6.1 that, for any $h > 0$, T_h admits a compact attractor \mathcal{A}_h which attracts bounded sets of X_h and is contained in $B_X(R_0) \cap X_h$, where R_0 is a constant independent of h . Note that R_0 can be chosen so that \mathcal{A} is also contained in $B_X(R_0)$.

(2) Now we check that, for any $r > 0$, there exists a constant $L(r) > 0$ such that, for all v and w in the ball $\bar{B}(r) = \{v \in H_0^1(\Omega) : \|v\|_{H_0^1(\Omega)} \leq r\}$, we have

$$(6.22) \quad \|f(v) - f(w)\|_{H^{-1}(\Omega)} \leq L(r)\|v - w\|_{L^2(\Omega)}.$$

Indeed, we can write

$$\begin{aligned} \|f(v) - f(w)\|_{H^{-1}(\Omega)} &= \sup_{\Phi \in H_0^1(\Omega)} \frac{\int_{\Omega} (f(v(x)) - f(w(x)))\Phi(x) \, dx}{\|\Phi\|_{H_0^1(\Omega)}} \\ &\leq \sup_{\Phi \in H_0^1(\Omega)} \frac{\int_{\Omega} \int_0^1 f'(w(x) + \tau(v(x) - w(x)))(v(x) - w(x))\Phi(x) \, dx \, d\tau}{\|\Phi\|_{H_0^1(\Omega)}}. \end{aligned}$$

Hence, using the hypothesis (6.2), we obtain

$$(6.23) \quad \begin{aligned} &\|f(v) - f(w)\|_{H^{-1}(\Omega)} \\ &\leq \sup_{\Phi \in H_0^1(\Omega)} \frac{C_1}{\|\Phi\|_{H_0^1(\Omega)}} \left\{ \left(\int_{\Omega} |v(x) - w(x)|^2 \, dx \right)^{1/2} \right. \\ &\quad \times \left(\int_{\Omega} 2(|v(x)|^\beta + |w(x)|^\beta + 1) \, dx \right) \\ &\quad \left. \times \left(\int_{\Omega} |\Phi(x)|^6 \, dx \right)^{1/6} \right\}, \end{aligned}$$

where $\beta = \sup(3, 6 - 3\gamma)$. As $H_0^1(\Omega) \hookrightarrow L^6(\Omega)$, the property (6.22) is a direct consequence of (6.23).

(3) Now, for any $t_1 > 0$, we estimate $\|(u(t) - u_h(t), \partial u(t)/\partial t - \partial u_h(t)/\partial t)\|_Y$ for $0 \leq t \leq t_1$, where $u(t)$ and $u_h(t)$ are the solutions of the equations (6.1) and (6.1)_h, respectively, with initial condition $(\phi_h, \psi_h) \in B_X(R_0)$. Thanks to the hypothesis (6.21)(ii), we have, on the one hand,

$$(6.24) \quad \|u(t) - Q_h u(t)\|_{L^2(\Omega)} \leq K_1 h^\theta C_1(R_0),$$

and, on the other hand,

$$\begin{aligned} \left\| \frac{\partial u}{\partial t}(t) - \frac{\partial}{\partial t} Q_h u(t) \right\|_{H^{-1}(\Omega)} &= \sup_{v \in H_0^1(\Omega)} \left[\frac{\partial u}{\partial t} - \frac{\partial}{\partial t} Q_h u, v - Q_h v \right] / \|v\|_{H_0^1(\Omega)} \\ &\leq \sup_{v \in H_0^1(\Omega)} \left\| \frac{\partial u}{\partial t} - \frac{\partial}{\partial t} Q_h u \right\|_{L^2(\Omega)} \frac{\|v - Q_h v\|_{L^2(\Omega)}}{\|v\|_{H_0^1(\Omega)}}, \end{aligned}$$

which gives

$$(6.25) \quad \left\| \frac{\partial u}{\partial t}(t) - \frac{\partial}{\partial t} Q_h u(t) \right\|_{H^{-1}(\Omega)} \leq 2K_1 h^\theta C_1(R_0).$$

It remains to estimate the term $\|(Q_h u(t) - u_h(t), \partial Q_h u(t)/\partial t - \partial u_h(t)/\partial t)\|_Y$ for $0 \leq t \leq t_1$. Note that by (6.21)(i), Q_h can be extended to a continuous, linear operator from $H^{-1}(\Omega)$ into V_h and that the element $u_h - Q_h u$ thus satisfies the equation

$$(6.26) \quad \begin{aligned} \frac{\partial^2}{\partial t^2}(u_h - Q_h u) + 2\alpha \frac{\partial}{\partial t}(u_h - Q_h u) + A_h(u_h - Q_h u) \\ = -Q_h(f(u_h) - f(u)) - (A_h Q_h - Q_h A)u. \end{aligned}$$

We now introduce the operator $S_h \in \mathcal{L}(H^{-1}(\Omega); V_h)$ given by

$$(6.27) \quad \forall f \in H^{-1}(\Omega), \quad a(S_h f, v_h) = [f, v_h], \quad \forall v_h \in V_h.$$

Clearly, one has

$$(6.28) \quad \|S_h f\|_{H_0^1(\Omega)} \leq c \|f\|_{H^{-1}(\Omega)},$$

where $c > 0$ is a constant independent of h .

Taking the inner product in $L^2(\Omega)$ of (6.26) by $S_h(\partial(u_h - Q_h u)/\partial t)$ and using the relation (6.27), we obtain

$$(6.29) \quad \begin{aligned} a \left(\frac{\partial^2}{\partial t^2} S_h(u_h - Q_h u), \frac{\partial}{\partial t} S_h(u_h - Q_h u) \right) \\ + 2\alpha a \left(S_h \frac{\partial}{\partial t}(u_h - Q_h u), S_h \frac{\partial}{\partial t}(u_h - Q_h u) \right) \\ + a \left(u_h - Q_h u, S_h \frac{\partial}{\partial t}(u_h - Q_h u) \right) \\ = - \left[f(u_h) - f(u), S_h \frac{\partial}{\partial t}(u_h - Q_h u) \right] \\ + a \left(u - Q_h u, S_h \frac{\partial}{\partial t}(u_h - Q_h u) \right). \end{aligned}$$

But

$$\begin{aligned} a \left(u - Q_h u, S_h \frac{\partial}{\partial t}(u_h - Q_h u) \right) &= a \left(P_h u - Q_h u, S_h \frac{\partial}{\partial t}(u_h - Q_h u) \right) \\ &= \left[P_h u - Q_h u, \frac{\partial}{\partial t}(u_h - Q_h u) \right] = \left[P_h u - u, \frac{\partial}{\partial t}(u_h - Q_h u) \right] \end{aligned}$$

and

$$a \left(u_h - Q_h u, S_h \frac{\partial}{\partial t}(u_h - Q_h u) \right) = \left[u_h - Q_h u, \frac{\partial}{\partial t}(u_h - Q_h u) \right].$$

Then, from (6.29) we can derive the following inequality:

$$\begin{aligned}
 & \frac{1}{2} \frac{\partial}{\partial t} \left\| \frac{\partial}{\partial t} S_h(u_h - Q_h u) \right\|_{H_0^1(\Omega)}^2 + 2\alpha \left\| \frac{\partial}{\partial t} S_h(u_h - Q_h u) \right\|_{H_0^1(\Omega)}^2 \\
 & \quad + \frac{1}{2} \frac{\partial}{\partial t} \|u_h - Q_h u\|_{L^2(\Omega)}^2 \\
 & \leq \|f(u) - f(u_h)\|_{H^{-1}(\Omega)} \left\| \frac{\partial}{\partial t} S_h(u_h - Q_h u) \right\|_{H_0^1(\Omega)} \\
 & \quad + \|u - P_h u\|_{L^2(\Omega)} \left\| \frac{\partial}{\partial t} (u_h - Q_h u) \right\|_{L^2(\Omega)}.
 \end{aligned}$$

Using the property (6.22) and the fact that $(u, \partial u / \partial t)$ and $(u_h, \partial u_h / \partial t)$ belong to $B_X(C_1(R_0))$, we infer from the above estimate that

$$\begin{aligned}
 (6.30) \quad & \frac{\partial}{\partial t} \left\| \frac{\partial}{\partial t} S_h(u_h - Q_h u) \right\|_{H_0^1(\Omega)}^2 + \frac{\partial}{\partial t} \|u_h - Q_h u\|_{L^2(\Omega)}^2 \\
 & \leq \frac{L^2(C_1(R_0))}{\alpha} \left\{ \|u - Q_h u\|_{L^2(\Omega)}^2 + \|u_h - Q_h u\|_{L^2(\Omega)}^2 \right\} \\
 & \quad + 2C_1(R_0) \|u - P_h u\|_{L^2(\Omega)}.
 \end{aligned}$$

Integrating (6.30) from 0 to t and using Gronwall's lemma as well as the hypothesis (6.21)(ii), we get, for $0 \leq t \leq t_1$,

$$(6.31) \quad \left\| \frac{\partial}{\partial t} S_h(u_h - Q_h u)(t) \right\|_{H_0^1(\Omega)}^2 + \|(u_h - Q_h u)(t)\|_{L^2(\Omega)}^2 \leq K_2 t_1 e^{K_3 t_1} h^\theta,$$

where $K_2 > 0$ and $K_3 > 0$ are two constants depending on R_0 only. Now we remark that

$$\begin{aligned}
 \left\| \frac{\partial}{\partial t} (u_h - Q_h u) \right\|_{H^{-1}(\Omega)} &= \sup_{v \in H_0^1(\Omega)} \left[\frac{\partial}{\partial t} (u_h - Q_h u), v \right] / \|v\|_{H_0^1(\Omega)} \\
 &= \sup_{v \in H_0^1(\Omega)} \left[\frac{\partial}{\partial t} (u_h - Q_h u), Q_h v \right] / \|v\|_{H_0^1(\Omega)} \\
 &= \sup_{v \in H_0^1(\Omega)} a \left(\frac{\partial}{\partial t} S_h(u_h - Q_h u), Q_h v \right) / \|v\|_{H_0^1(\Omega)}
 \end{aligned}$$

and therefore, thanks to the hypothesis (6.21)(i),

$$(6.32) \quad \left\| \frac{\partial}{\partial t} (u_h - Q_h u) \right\|_{H^{-1}(\Omega)} \leq K_0 \left\| \frac{\partial}{\partial t} S_h(u_h - Q_h u) \right\|_{H_0^1(\Omega)}.$$

Finally, by (6.24), (6.25), (6.31) and (6.32), we obtain, for $0 \leq t \leq t_1$,

$$(6.33) \quad \left\| (u(t) - u_h(t), \frac{\partial u}{\partial t}(t) - \frac{\partial u_h}{\partial t}(t)) \right\|_Y \leq K_4 t_1^{1/2} e^{K_5 t_1} h^{\theta/2},$$

where K_4 and K_5 are positive constants depending on R_0 only.

(4) Since, for any $h > 0$, $\mathcal{A}_h \subset B_X(R_0)$, we deduce from the property (6.33), by arguing as in the proof of Proposition 2.10 (or in Remark 2.7), that, for any $\varepsilon_0 > 0$, there exists $h_0 > 0$ such that, for $h \leq h_0$, $\delta_Y(\mathcal{A}_h, \mathcal{A}) \leq \varepsilon_0$. \square

Remark 6.2. The results of Theorems 6.1 and 6.2 extend easily to the cases where Ω is a bounded domain in \mathbf{R} or \mathbf{R}^2 (for the conditions on f , see Babin and Vishik [1] or Hale [12]).

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